THE SEARCH FOR THE SMALLEST 3-E.C. GRAPHS

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Theorem
(This is a joint work with Paweł Prałat.)

Adjacency properties of graphs, first studied by Erdős and Rényi, have received much attention. One such property is the n-e.c. property. For a positive integer n, a graph is *n-existentially closed* or *n-e.c.*, if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B can be empty), there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. We say that z is *correctly joined* to A and B. For completeness, every graph is 0-e.c.

Because for every positive integer n there is a graph that satisfies n-e.c. property, therefore we can define $m_{ec}(n)$ to be the minimum order of an n-e.c. graph. It is known that $m_{ec}(1) = 4$ and $m_{ec}(2) = 9$, but no other values of this function are determined. For example, $20 \leq m_{ec}(3) \leq 28$. Similar to the well-known Ramsey numbers, it is difficult to compute the exact value of $m_{ec}(n)$, even for $n = 3$, and very little progress has been made.

During my talk, I would like to present a few initial results that improve the lower bound, namely, $m_{ec}(3) \geq 23$. An initial results also *suggest* that there is no 3-e.c. graph of order 23 and I hope to be able to announce this during my talk. In order to obtain the result, a computer support was required to check that no graph satisfies a necessary condition described in the following theorem.

Theorem 1. *Let $n \geq m \geq 1$. If G is n-e.c., then for all disjoint sets of vertices X and Y with $|X \cup Y| = m$ (one of X or Y can be empty), a graph induced by vertex set $Z = Z(X, Y)$ defined as*

$$Z = \left(\bigcap_{x \in X} N(x) \right) \cap \left(\bigcap_{y \in Y} N^c(y) \right)$$

is $(n - m)$-e.c.

The main idea is based on this theorem but an additional interesting and subtle approaches were required to obtain the result.

During my talk, I would like to also describe a few open problems in this field, including the following intriguing one. From the above theorem we have that $m_{ec}(n) \geq 2m_{ec}(n-1)+1$. As $m_{ec}(3) \geq 23$ we derive that $m_{ec}(n) \geq 3 \cdot 2^n - 1$. On the other hand, using the random graph $G(v, 1/2)$ one has that $m_{ec}(n) = O(n^2 2^n)$. The most important open problem in this area is to determine whether $\lim_{n\to\infty} \frac{m_{ec}(n)}{2^n}$ exists and, if so to find its value.

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