Orientations of Planar Graphs

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Będlewo
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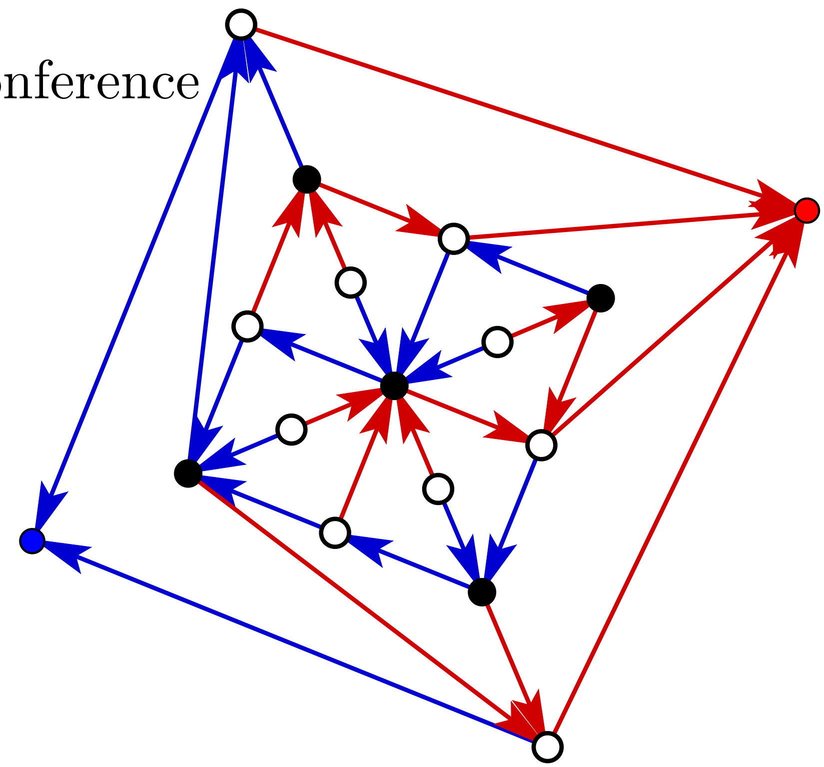

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Topics

α-Orientations

Sample Applications

Counting I: Estimates

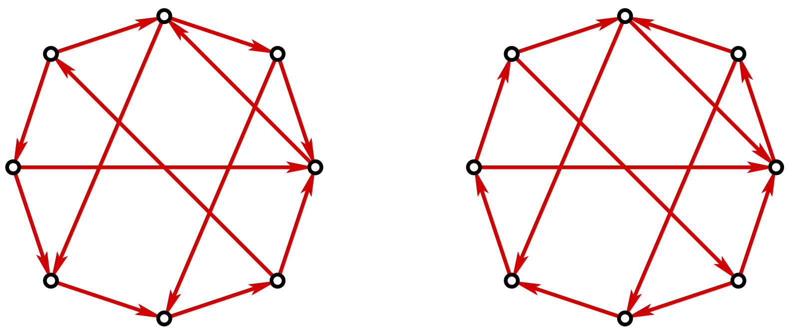
Counting II: Exact

Lattices

alpha-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with
$\text{outdeg}(v) = \alpha(v)$ for all v.

Example.



Two orientations for the same α.

alpha-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with
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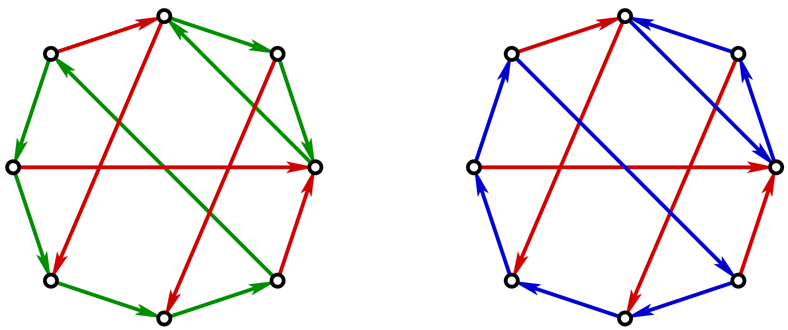
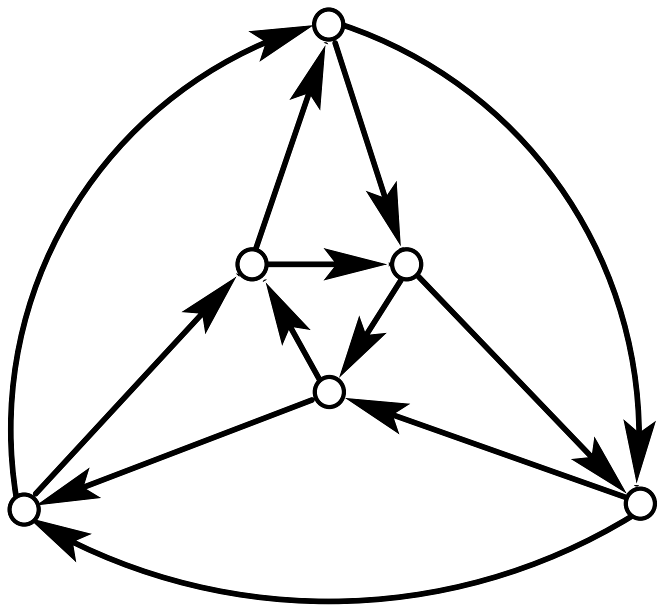


Two orientations for the same α.

Example 1: Eulerian Orientations

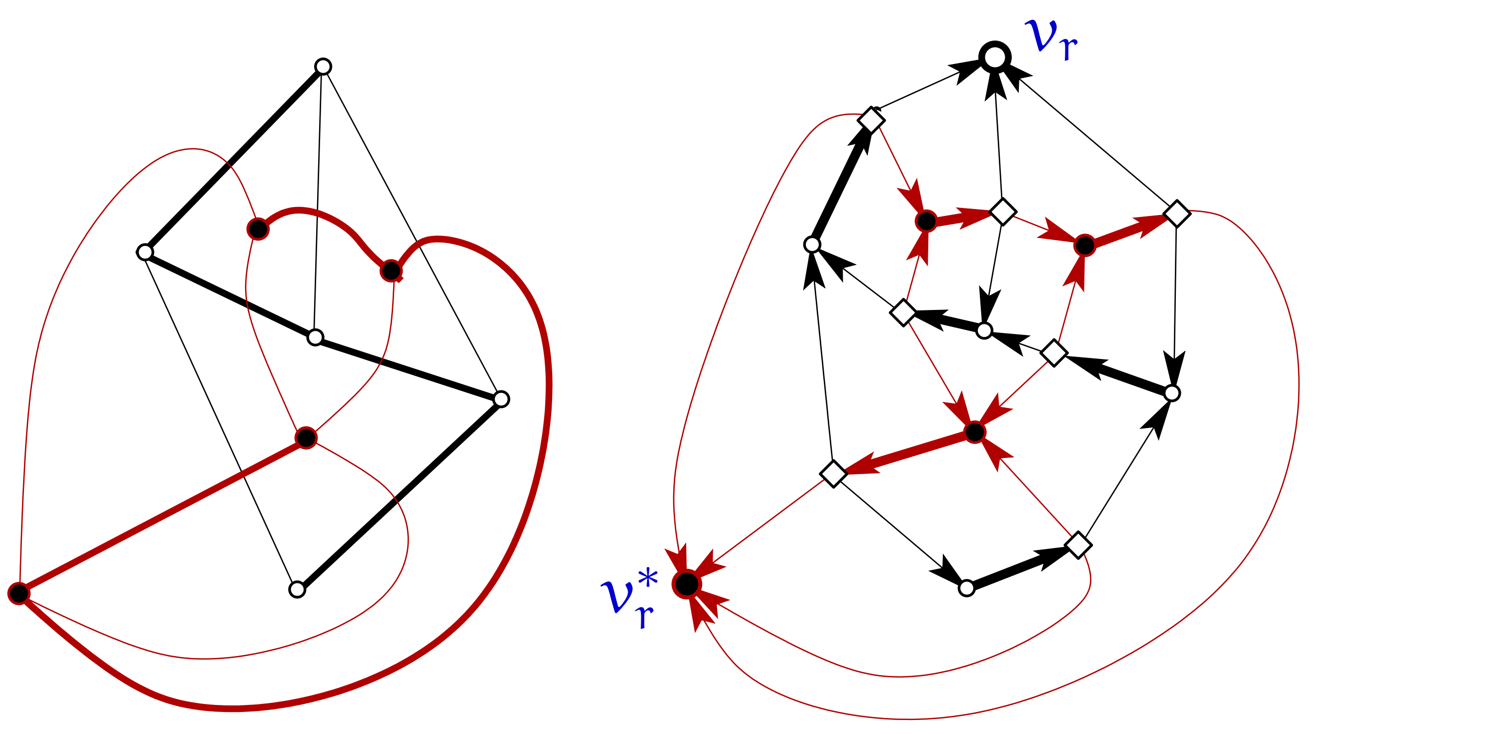
- Orientations with $\text{outdeg}(v) = \text{indeg}(v)$ for all v, i.e. $\alpha(v) = \frac{d(v)}{2}$



Example 2: Spanning Trees of Planar Graphs

G a planar graph. Spanning trees of G are in bijection with α_T orientations of a rooted primal-dual completion $\widetilde{G}$

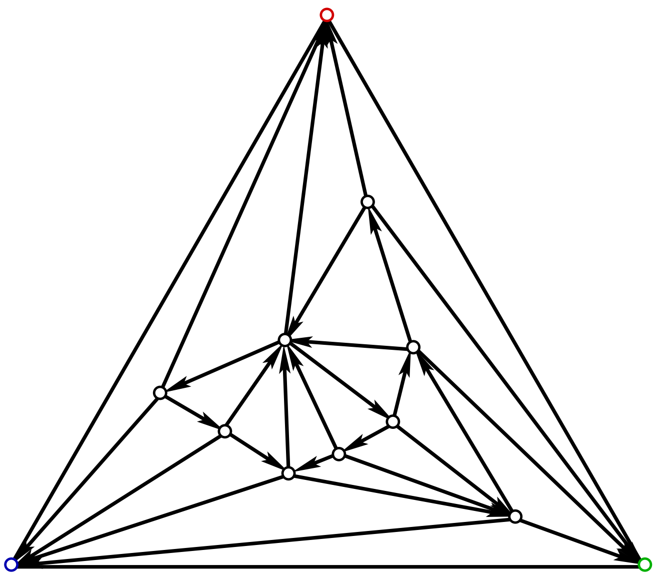
- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Example 3: 3-Orientations

G a planar triangulation, let

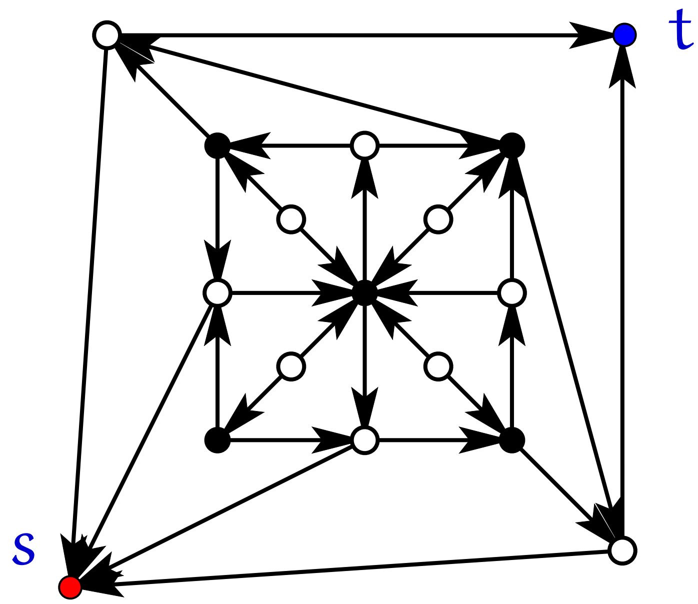
- $\alpha(v) = 3$ for each inner vertex and $\alpha(v) = 0$ for each outer vertex.



Example 4: 2-Orientations

G a planar quadrangulation, let

- $\alpha(v) = 0$ for an opposite pair of outer vertices and $\alpha(v) = 2$ for each other vertex.



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Sample Applications

Counting I: Estimates

Counting II: Exact

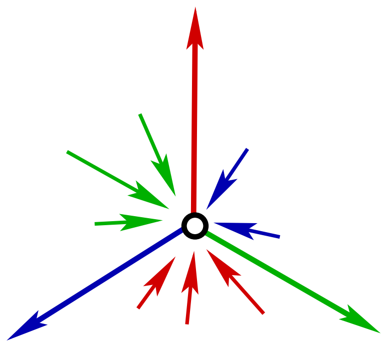
Lattices

Schnyder Woods

$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle.

A coloring and orientation of the interior edges of G with colors 1,2,3 is a Schnyder wood of G iff

- Inner vertex condition:



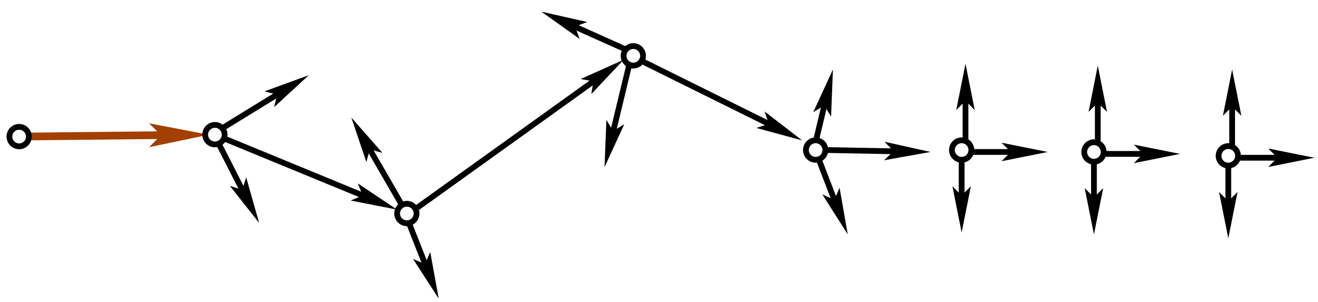
- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

Schnyder Woods and 3-Orientations

Theorem.
Schnyder woods and 3-orientations are equivalent.

Proof.

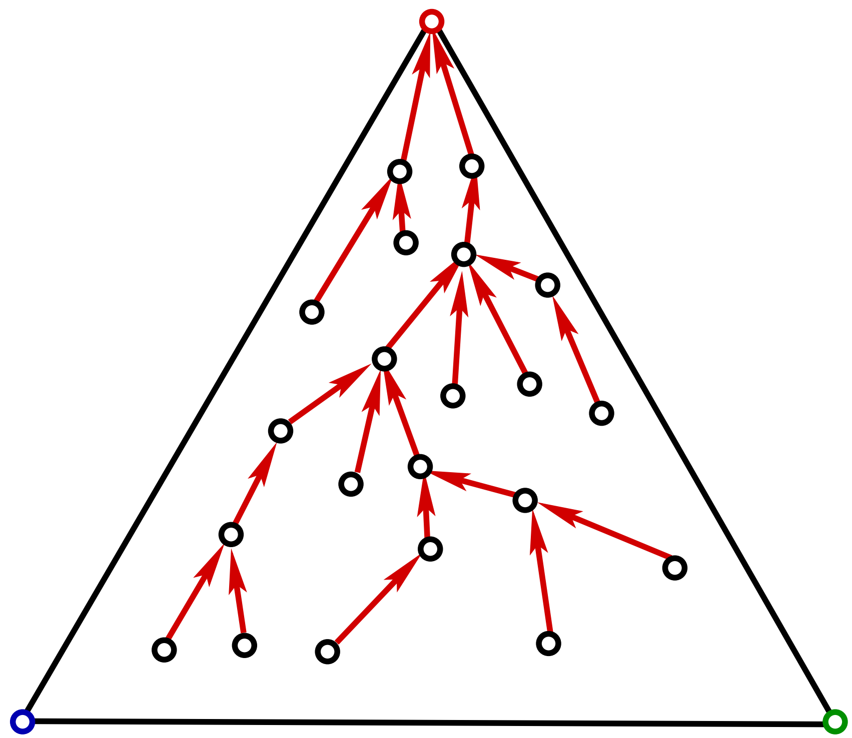
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i.

Schnyder Woods - Trees

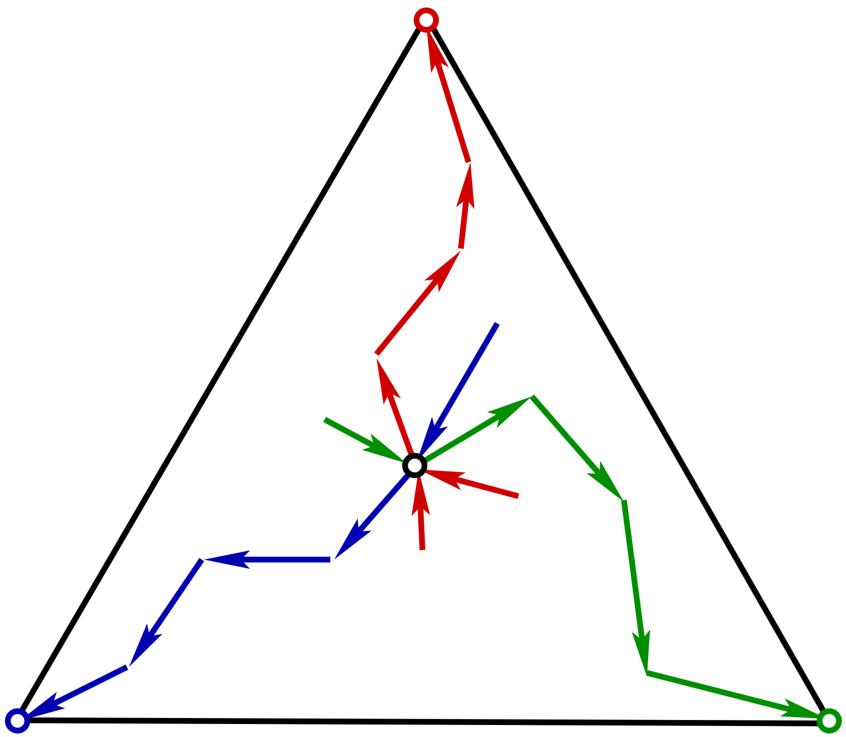
- The set T_i of edges colored i is a tree rooted at a_i.



Proof. Path $e \longrightarrow a_i$ is unique, c.f. 3-orientation.

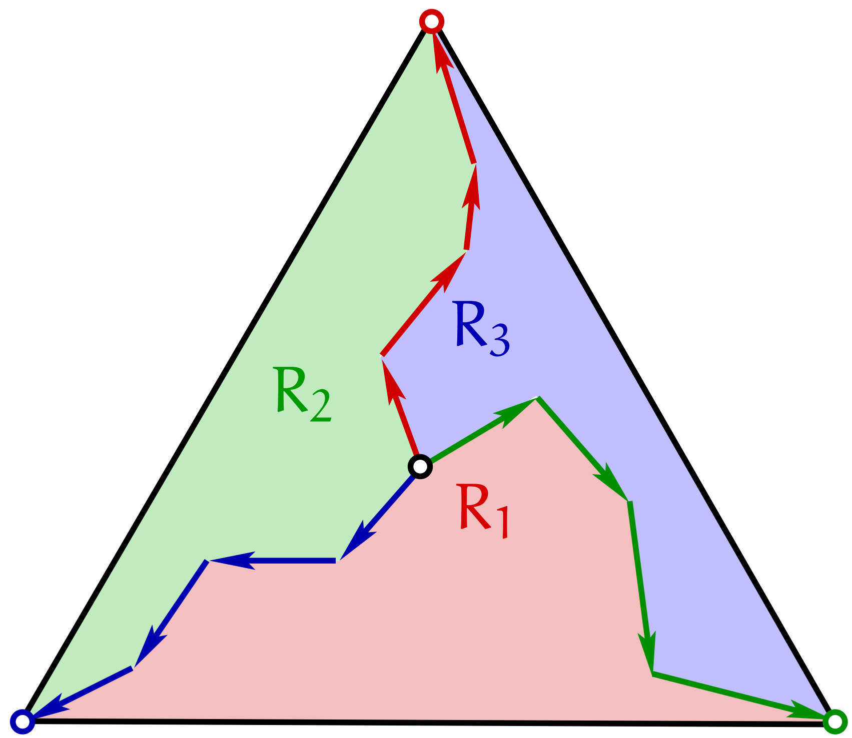
Schnyder Woods - Paths

- Paths of different color have at most one vertex in common.



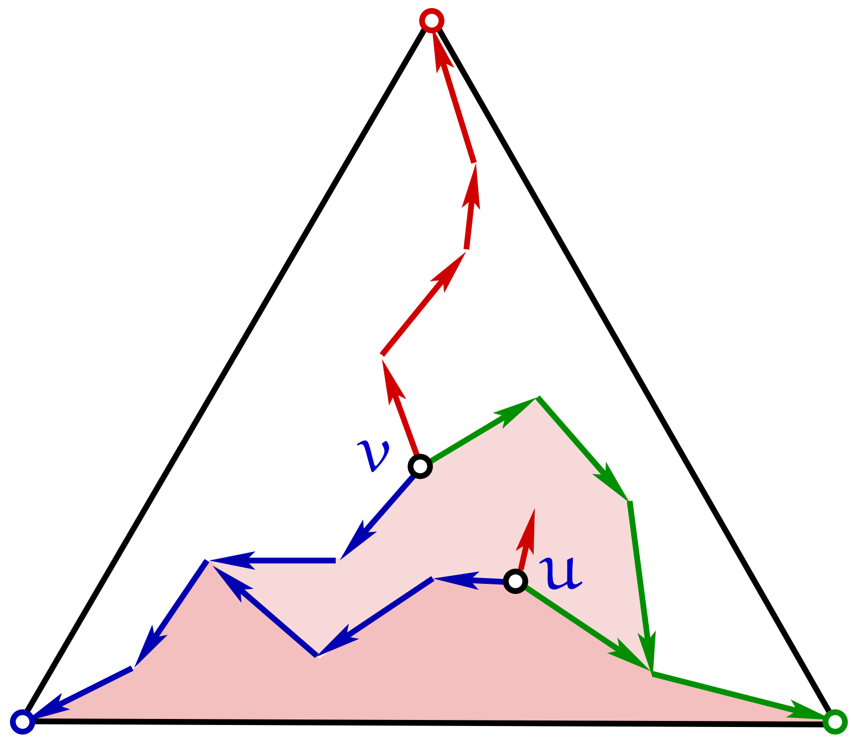
Schnyder Woods - Regions

- Every vertex has three distinguished regions.



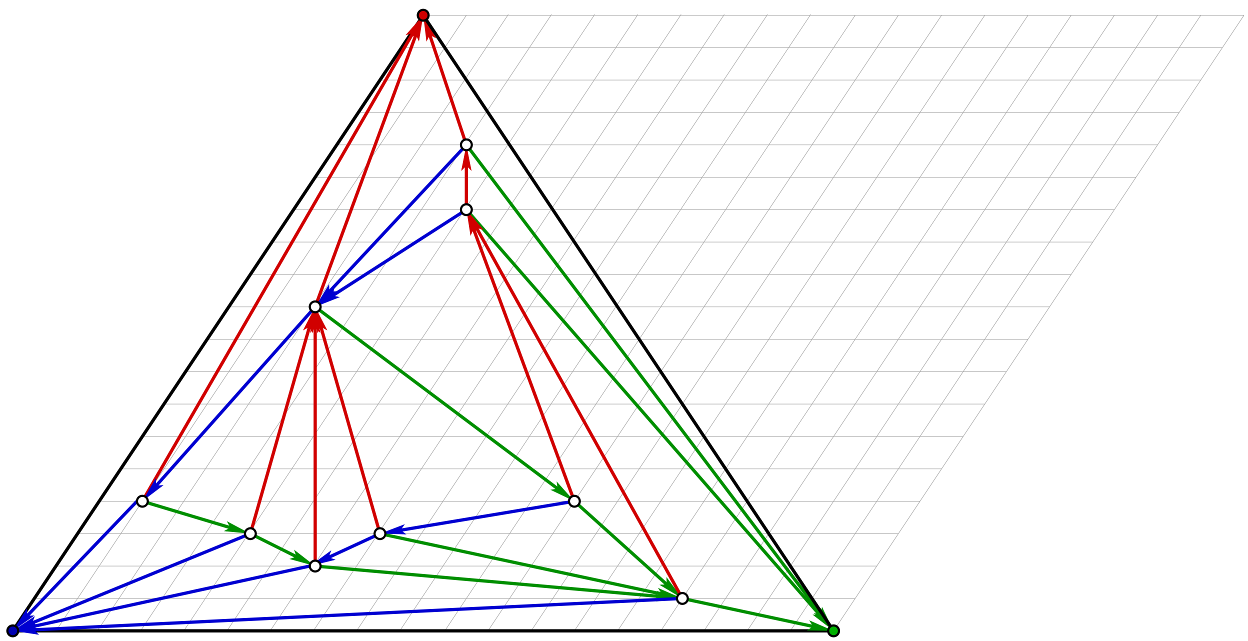
Schnyder Woods - Regions

- If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$.



Grid Drawings

The count of faces in the green and red region yields two coordinates (v_g, v_r) for vertex v.

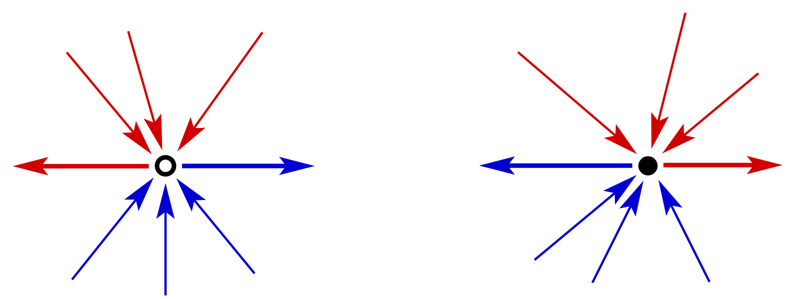


$\Longrightarrow$ straight line drawing on the $2n - 5 \times 2n - 5$ grid.

Separating Decompositions

$G = (V, E)$ a plane quadrangulation,
$F = \{a_0, x, a_1, y\}$ the outer face.

A coloring and orientation of the interior edges of G with colors 0, 1 is a separating decomposition of G iff

- Inner vertex condition: 

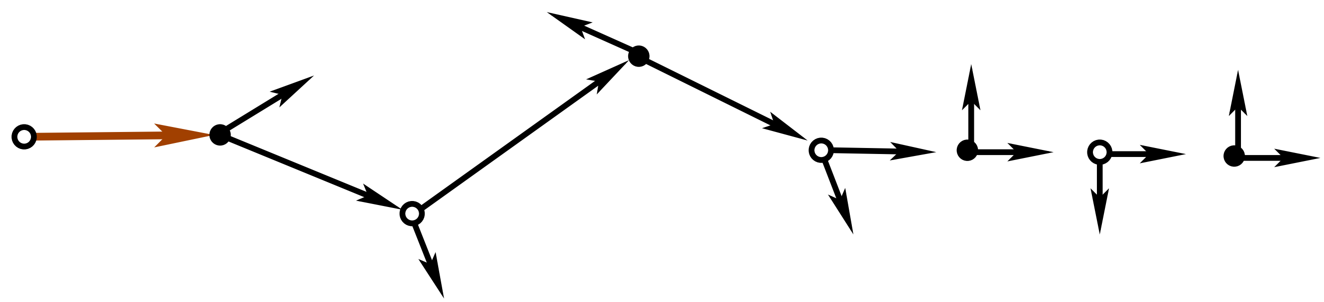
- Edges incident to a_0 and a_1 are oriented $v \to a_i$ in color i.

Separating Decompositions and 2-Orientations

Theorem.
Separating decompositions and 2-orientations are equivalent.

Proof.

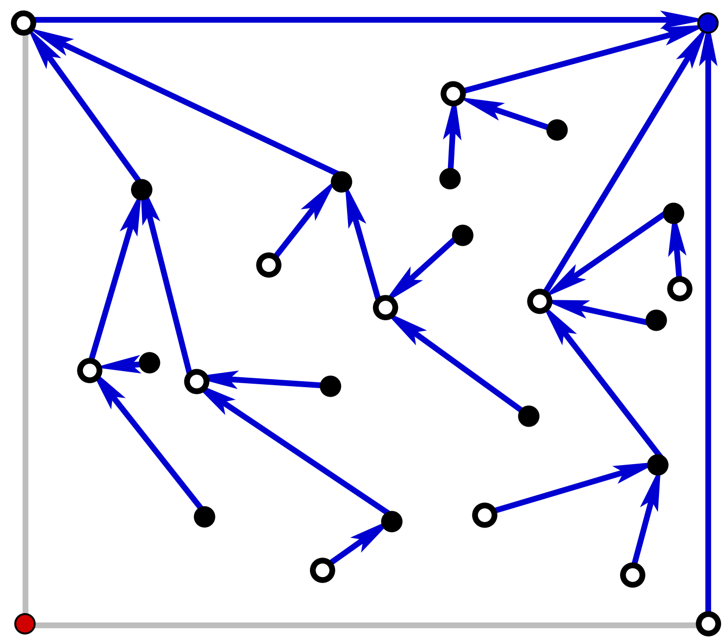
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i.

Separating Decompositions - Trees

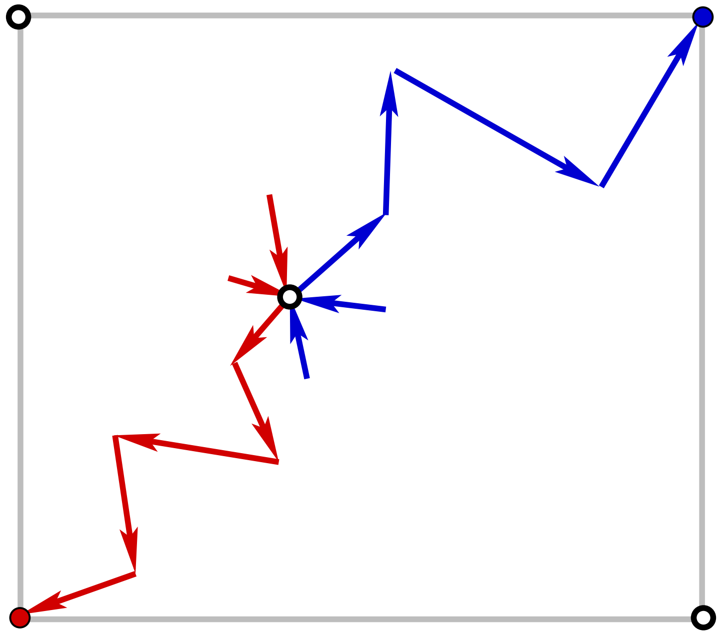
The set T_i of edges colored i is a tree rooted at a_i.



Proof. Path $e \longrightarrow a_i$ is unique, c.f. 2-orientation.

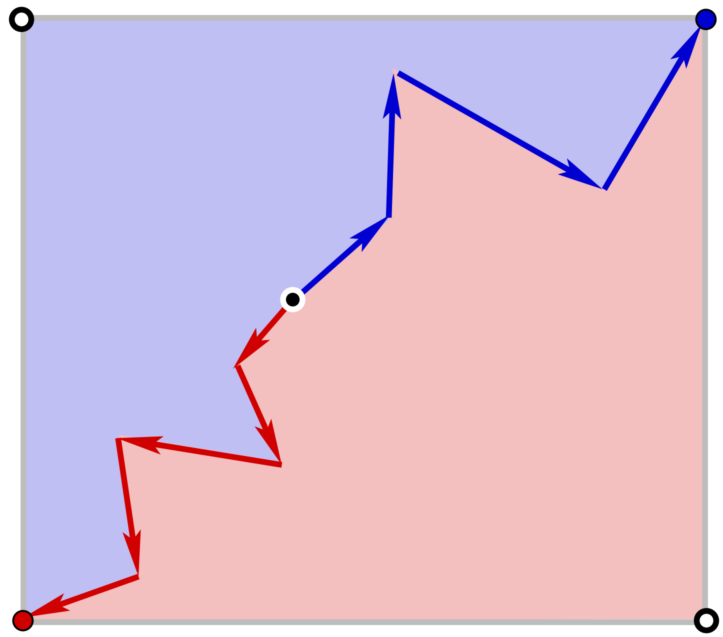
Separating Decompositions - Paths

- Paths of different color have at most one vertex in common.



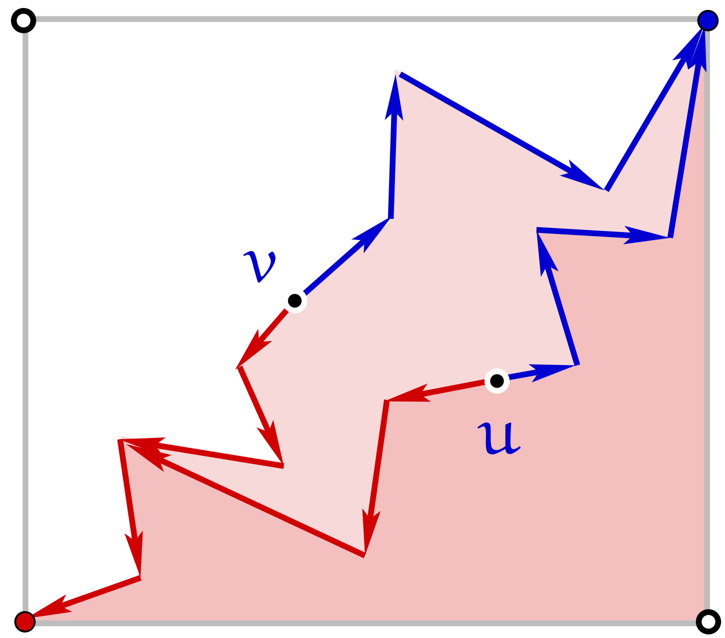
Separating Decompositions - Regions

- Every vertex has two distinguished regions.



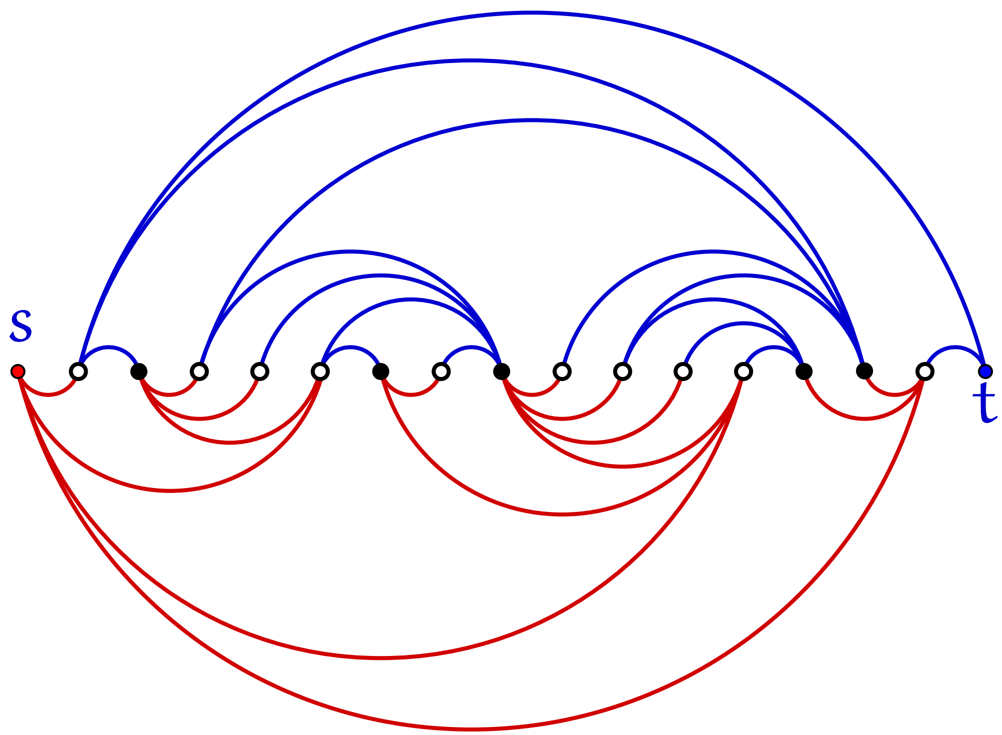
Separating Decompositions - Regions

- If $u \in R_0(v)$ then $R_0(u) \subset R_0(v)$.



2-Book Embedding

The count of faces in the red region yields a number ν_r for vertex $\nu \neq s, t$.



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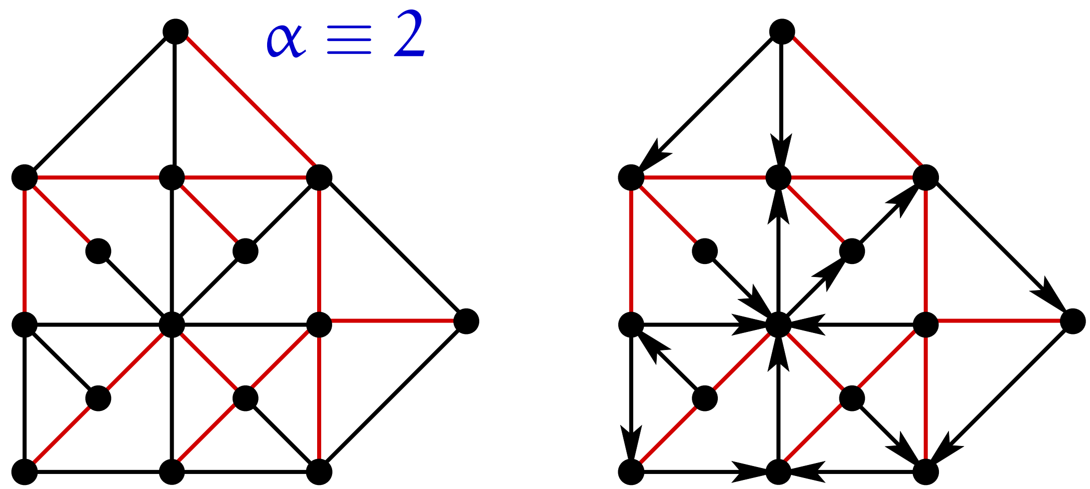
How Many?

Let G be a plane graph and $\alpha : V \longrightarrow \mathbb{N}$.

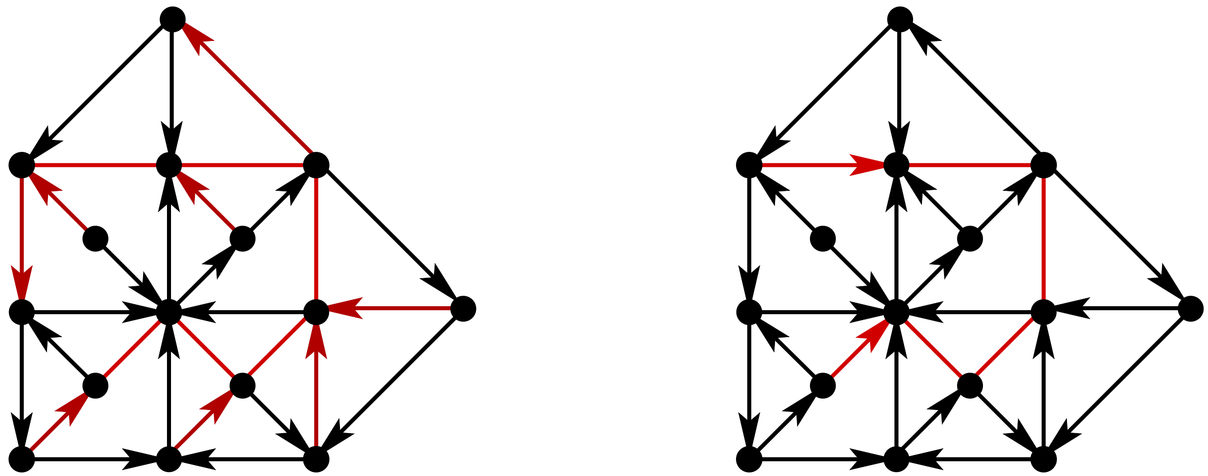
How many α-orientations can G have?

Choose a spanning tree T of G and orient the edges not in T randomly.



Towards an Upper Bound

If at all the orientation on $G - T$ is uniquely extendible.



$\Longrightarrow$ there are at most $2^{m-(n-1)}$ α-orientations.

Improve on one color

An orientation can be extended only if $\mathrm{outdeg}(v) \in \{\alpha(v), \alpha(v) - 1\}$ for all v.

Let I be an independent set of size $\geq \frac{n}{4}$ (Four Color Thm.)

Choose a tree T such that $I \subset \mathrm{leaves}(T)$.

Each $v \in I$ can independently obstruct extendability.

There are $\binom{d(v)-1}{\alpha(v)} + \binom{d(v)-1}{\alpha(v)-1} = \binom{d(v)}{\alpha(v)} \leq \binom{d(v)}{\lfloor d(v)/2 \rfloor}$ good choices for orientating edges at v.

The Result

Since

$$\mathrm{Prob}(d(v) = \alpha(v)) \leq \frac{1}{2^{d(v)-1}} \binom{d(v)}{\lfloor d(v)/2 \rfloor} \leq \frac{3}{4}$$

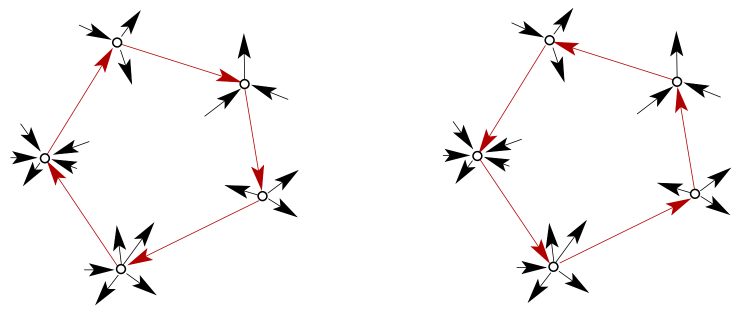
we conclude:

Theorem. The number of α-orientations of a plane graph on n vertices is at most

$$2^{m-n} \left(\frac{3}{4}\right)^{n/4} \leq 2^{2n} \left(\frac{3}{4}\right)^{n/4} \approx 3.73^n$$

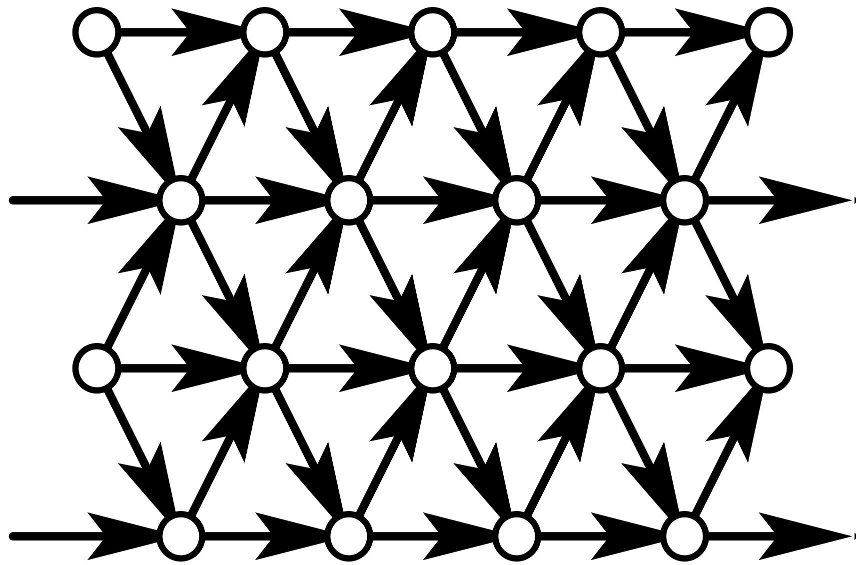
Towards a Lower Bound

Observation. Flipping cycles preserves α-orientations.

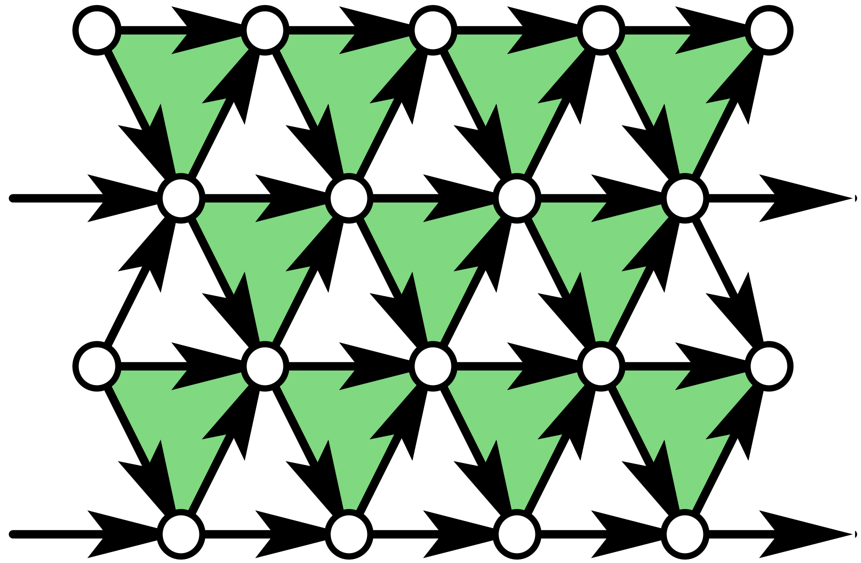


We show that there are many 3-orientation of the triangular lattice

The Initial Orientation

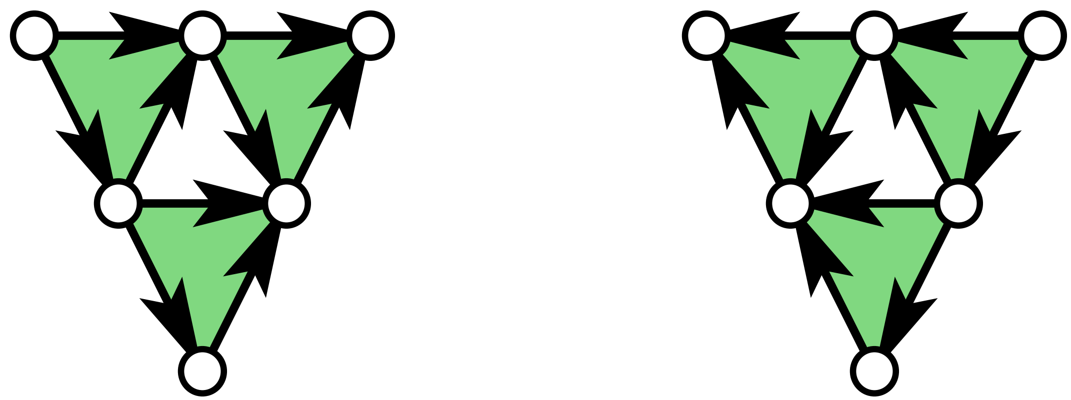


The Initial Orientation



Any subset of the green triangles can be flipped.

Green and White Flips



If 0 or 3 of the green neighbors are flipped a white triangle can be flipped.

using Jensen's ineq. $\Longrightarrow$

$\#$ 3-orientations $\geq 2^{\#\text{f}-\text{green}}\, 2^{\frac{2}{8}\#\text{f}-\text{white}} \approx 2^{\frac{5}{4}n} = 2.37^n.$

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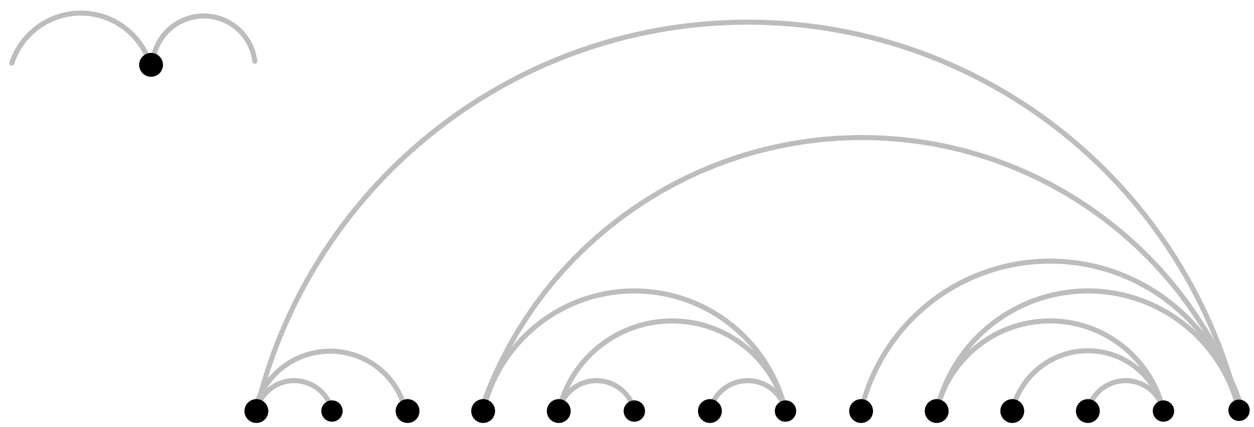
Alternating Layouts of Trees

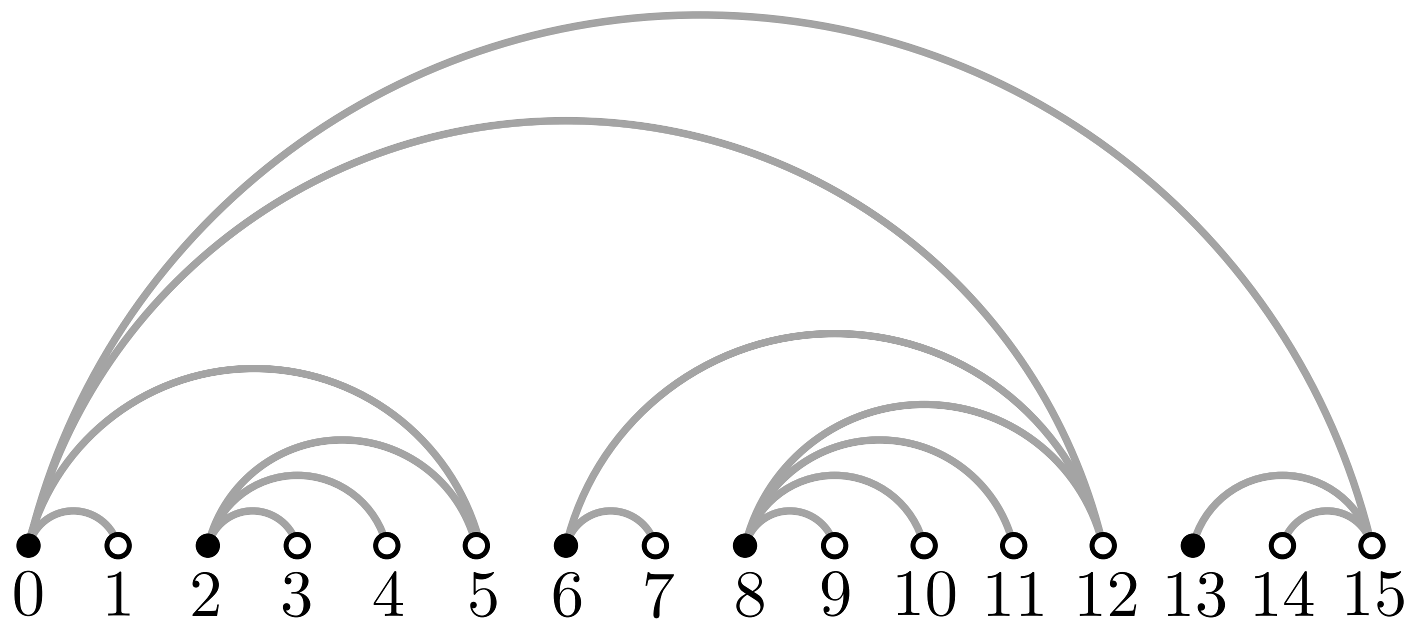
Definition. A numbering of the vertices of a tree is alternating if it is a 1-book embedding with no double-arc.

double arc



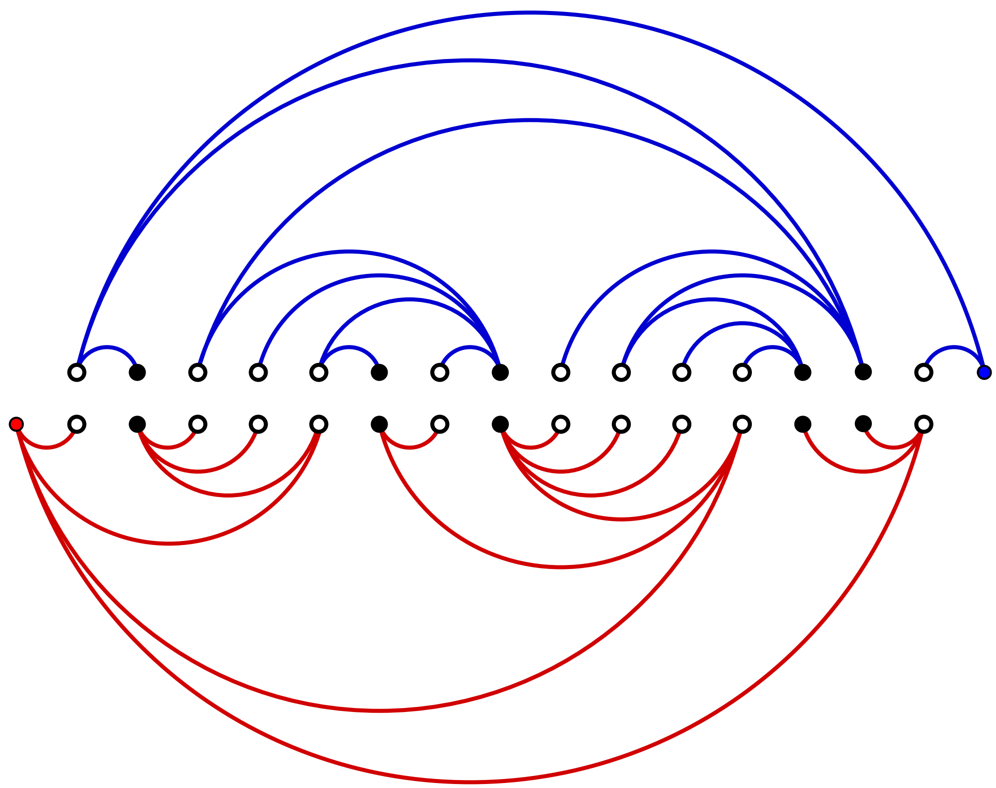
Alternating Layouts of Trees

Proposition. A rooted plane tree has a unique alternating layout with the root as leftmost vertex.



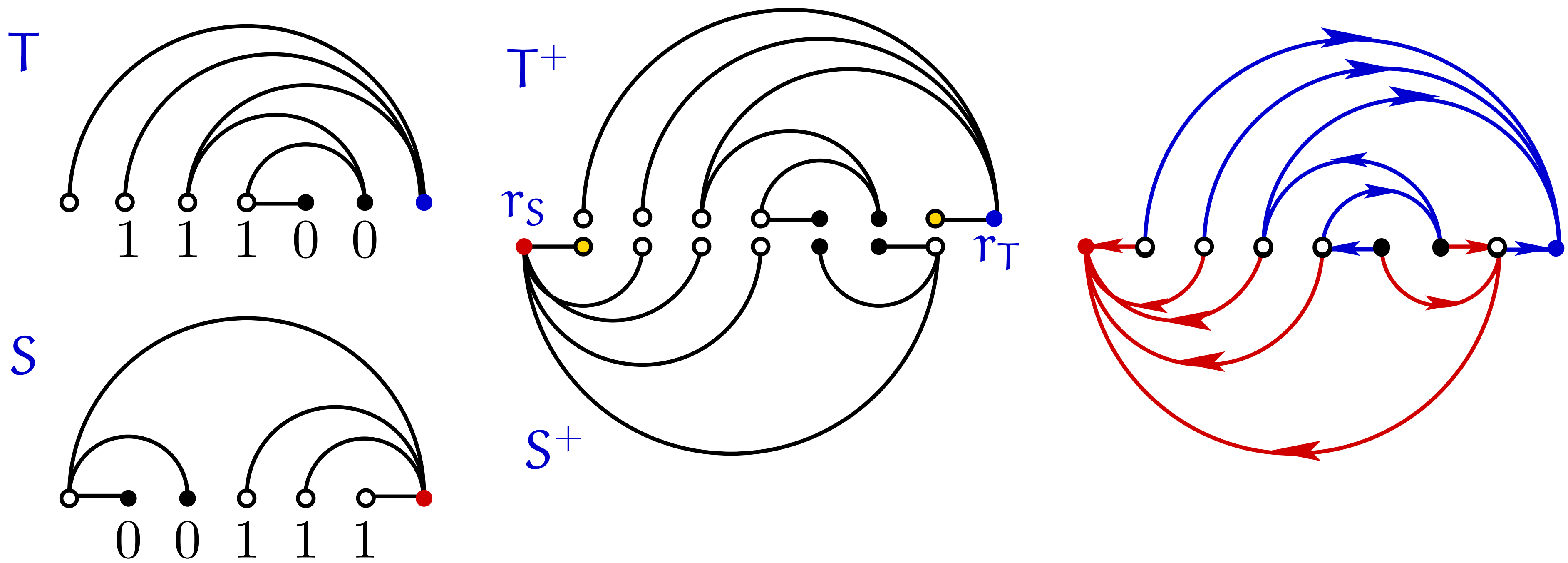
Separating Decompositions and Alternating Trees

Proposition. The 2-book embedding induced by a separation decomposition splits into two alternating trees.



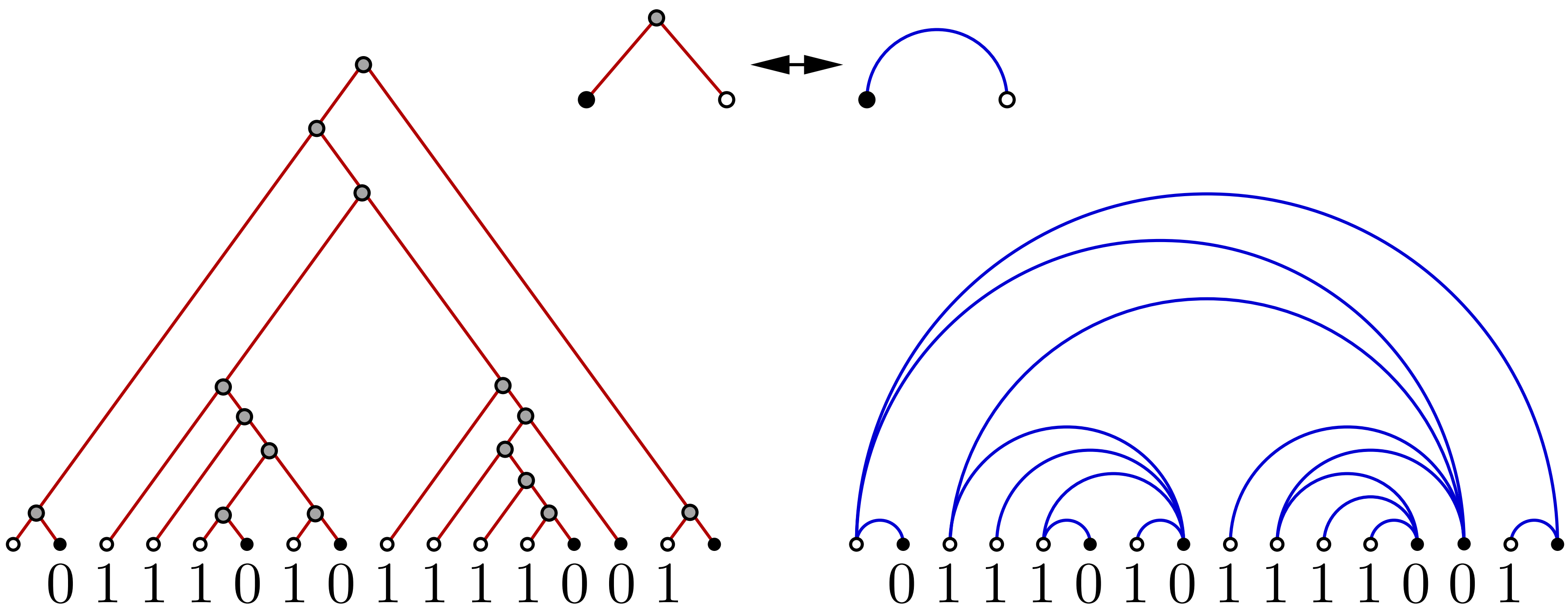
A Bijection

Theorem. There is a bijection between pairs (S, T) of alternating trees on n vertices with reverse fingerprints and separating decompositions of quadrangulations with $n+2$ vertices.



Alternating and Full* Binary Trees

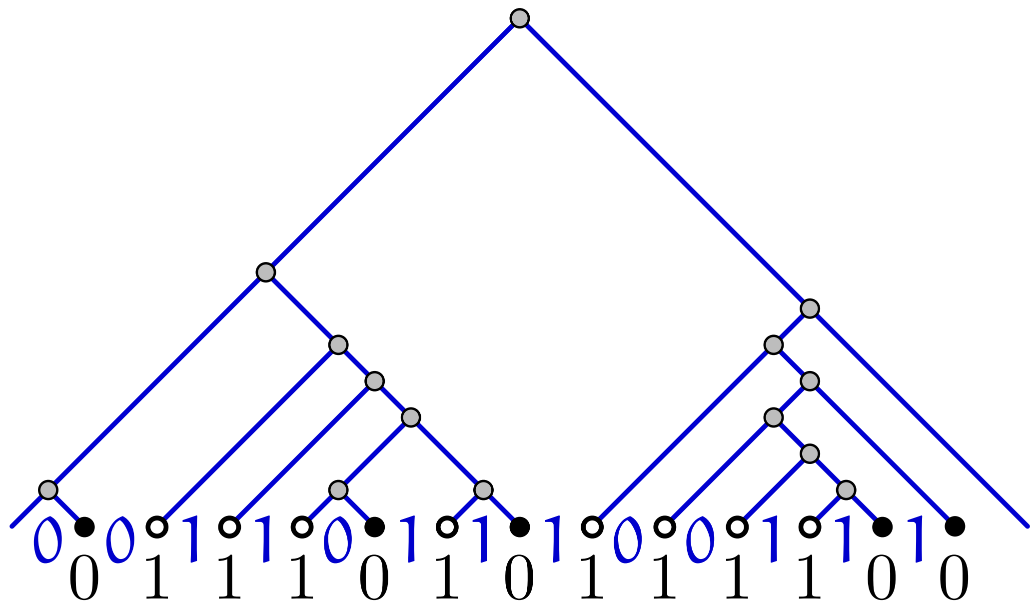
Proposition. There is bijection between alternating and binary trees that preserves fingerprints.



Encoding a Binary Tree

A 0-1 word α: Fingerprint.

A 0-1 word β: Inner nodes in in-order represented by 0 (left child) and 1 (right child) with the root being a 1 and omitting the last 1.

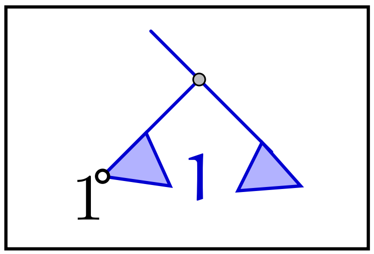


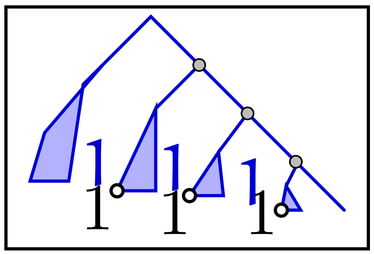
Encoding a Binary Tree, Cont.

A 0-1 word α: Fingerprint including the left extreme leaf.

A 0-1 word β: Inner nodes in in-order represented by 0 (left child) and 1 (right child) with the root being a 1.

Lemma. $\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \beta_i$ and $\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$





Lemma. The tree can be reconstructed.

Proof. The minimal k with $\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \beta_i$ and $\sum_{i=1}^{k+1} \alpha_i = \sum_{i=1}^{k+1} \beta_i$ determines the position of the root.

Counting Binary Trees

Proposition. The number of binary trees with $i+1$ left leaves and $j+1$ right leaves equals the number of nonintersecting lattice paths $\alpha'$and β' where:
$\alpha' : (0,1) \to (j, i+1)$
$\beta' : (1,0) \to (j+1, i)$

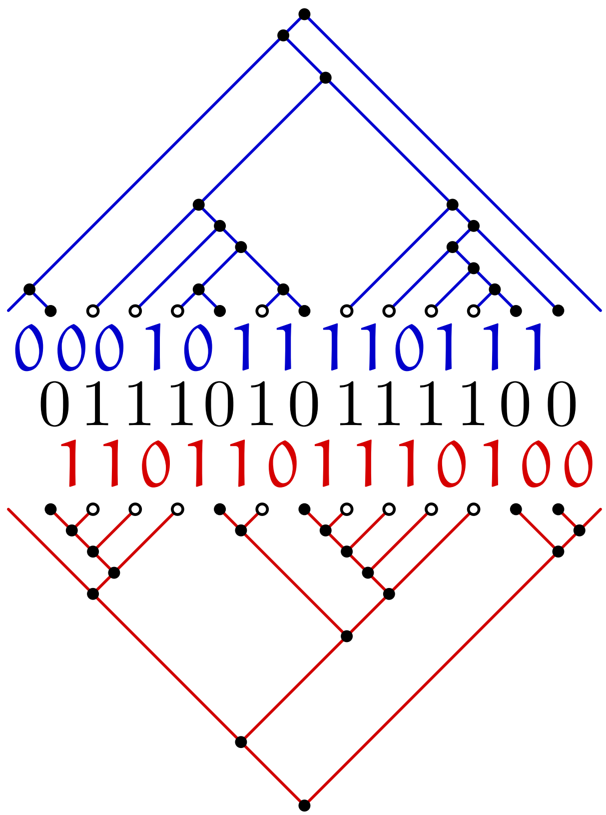
From the Lemma of Gessel Viennot we deduce that their number is

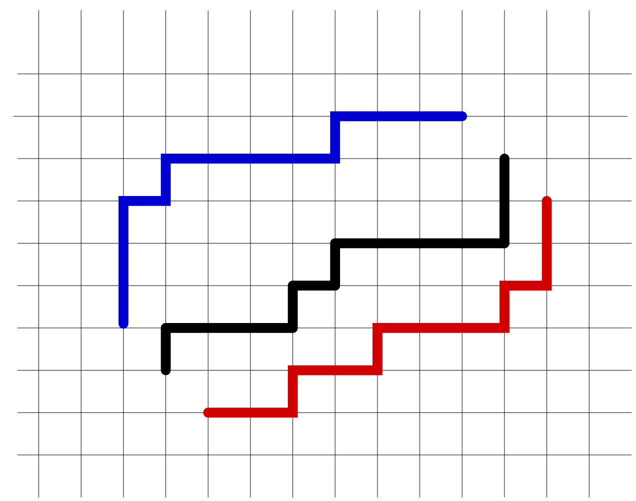
$$\det \begin{pmatrix} \binom{j+i}{j} & \binom{j+i}{j-1} \\ \binom{j+i}{j+1} & \binom{j+i}{j} \end{pmatrix} = \frac{1}{i+j+1}\binom{i+j+1}{j}\binom{i+j+1}{j+1}$$

This is the Narayana number $N(i+j+1, j)$.

Three Paths

Proposition. 2-orientations on $n+2$ vertices in total, $i+1$ of them white can be encoded by triples of disjoint lattice path.





Counting Baxter

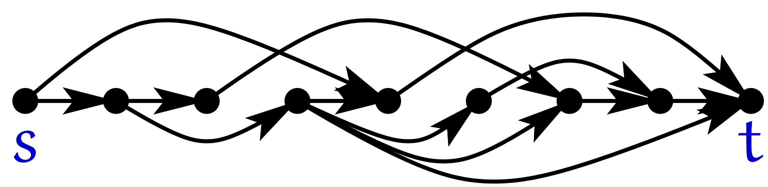
Theorem. The number of 2-orientations on $n+2$ vertices, separating decompositions .... is given by

$$\sum_{i=0}^{n-2} \frac{2n!(n-1)!(n-2)!}{i!(i+1)!(i+2)!(n-i)!(n-i-1)!(n-i-2)!} =$$

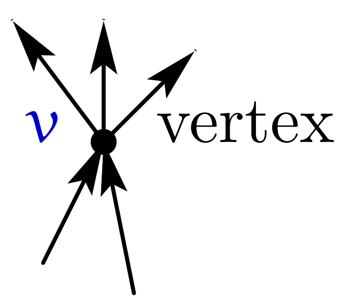
$$\frac{2}{n(n-1)^2} \sum_{i=0}^{n-2} \binom{n}{i} \binom{n}{i+1} \binom{n}{i+2}$$

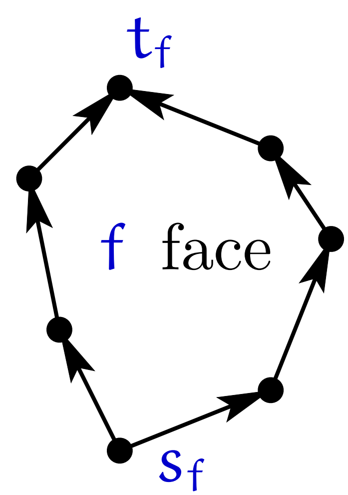
Bipolar Orientations

Definition. A bipolar orientation is an acyclic orientation with a unique source s and a unique sink t.



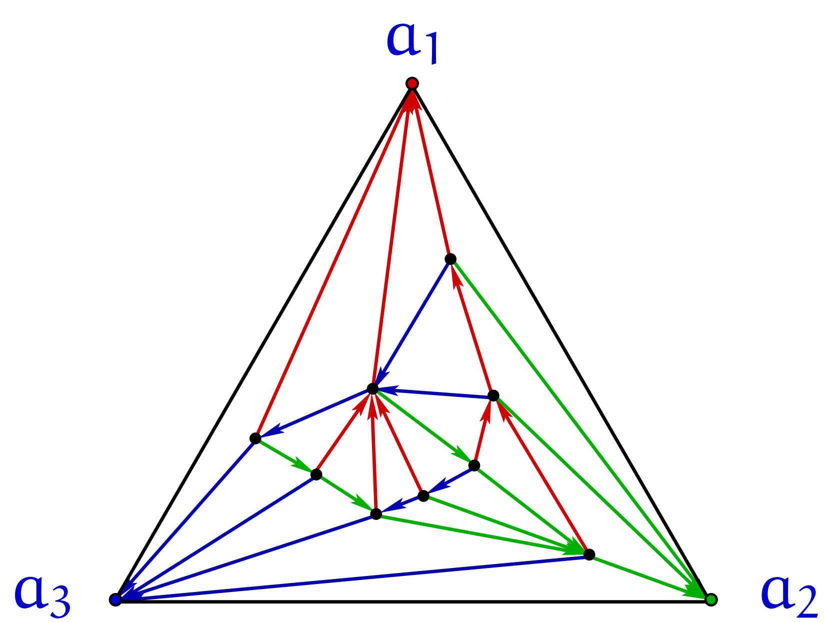
Plane bipolar orientations with s and t on the outer face are characterized by

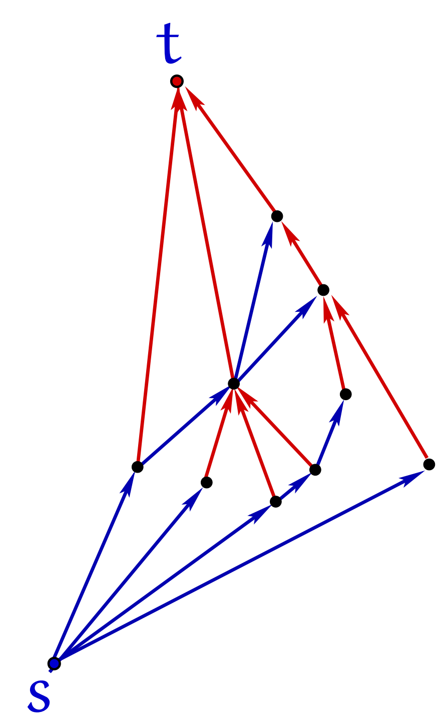




Schnyder Woods and Bipolar Orientations

Proposition. There is a bijection between Schnyder woods on triangulations with $n+3$ vertices and bipolar orientations of maps with $n+2$ vertices and the special property: $\star$ *The right side of every bounded face is of length two.*





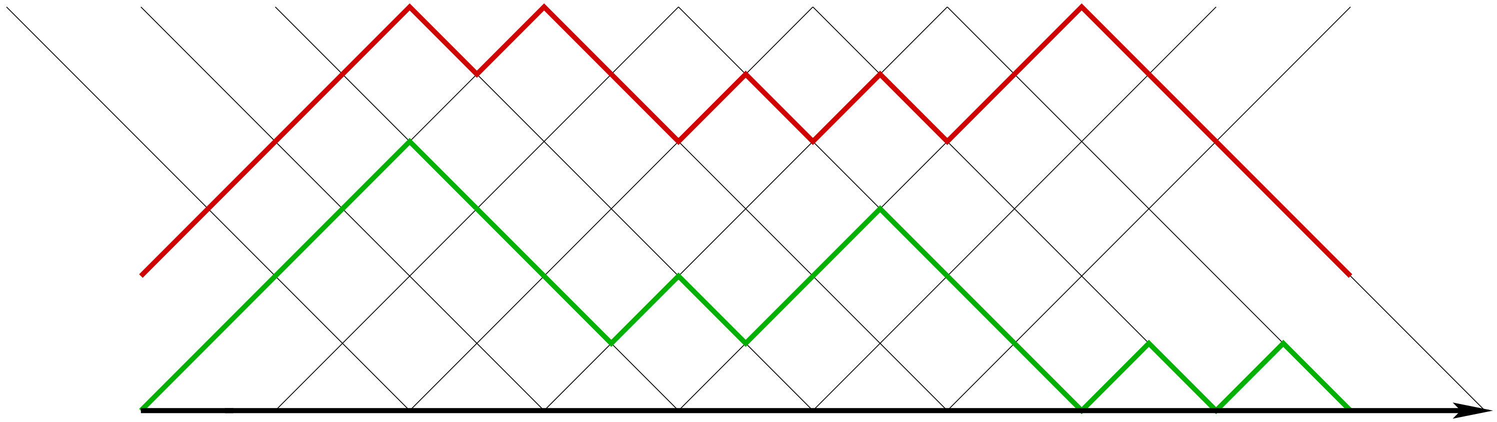
Special Property

Let T^b and T^r be the blue and red tree corresponding to a Schnyder wood. From $(\star')$ we get some crucial properties of the fingerprint and the bodyprints of the trees:

Fact. 1. Adding a leading 1 to the reduced fingerprint $\widehat{f}$, yields a Dyck word; in symbols $(01)^n \leq_{\mathrm{dom}} 1 + \widehat{f}$.

Fact. 2. The fingerprint uniquely determines the bodyprint of the blue tree, precisely $\overline{\beta^b} = 1 + \widehat{f}$.

Schnyder Woods and Dyck Path



Theorem [Bonichon]**.**
The number of Schnyder woods on plane triangulations on $n+3$ vertices equals the pairs of non-crossing Dyck-path of length $2n$ which is $C_{n+2}C_n - C_{n+1}^2$.

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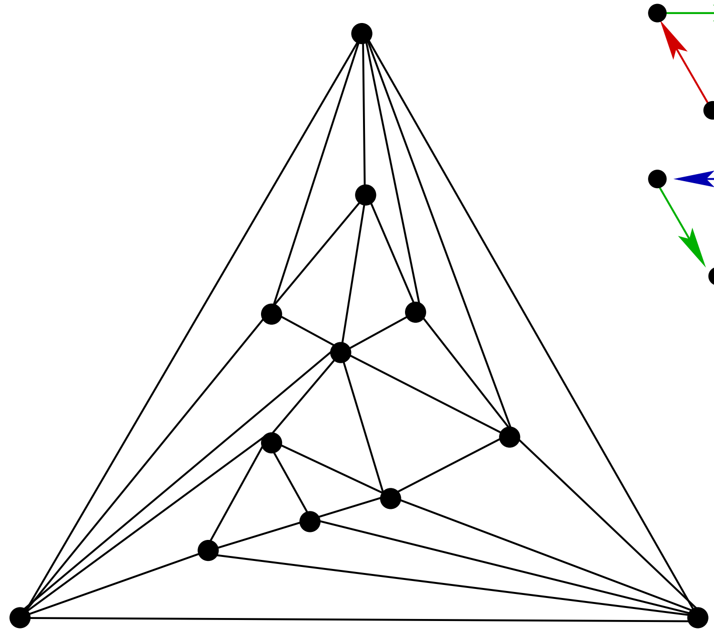
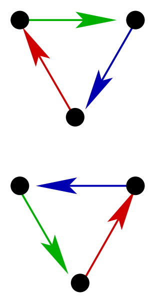
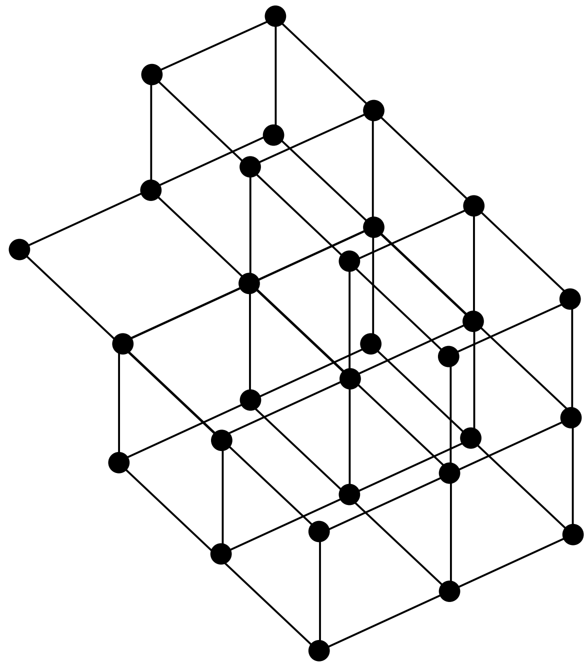
Distributive Lattices

Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

Example.

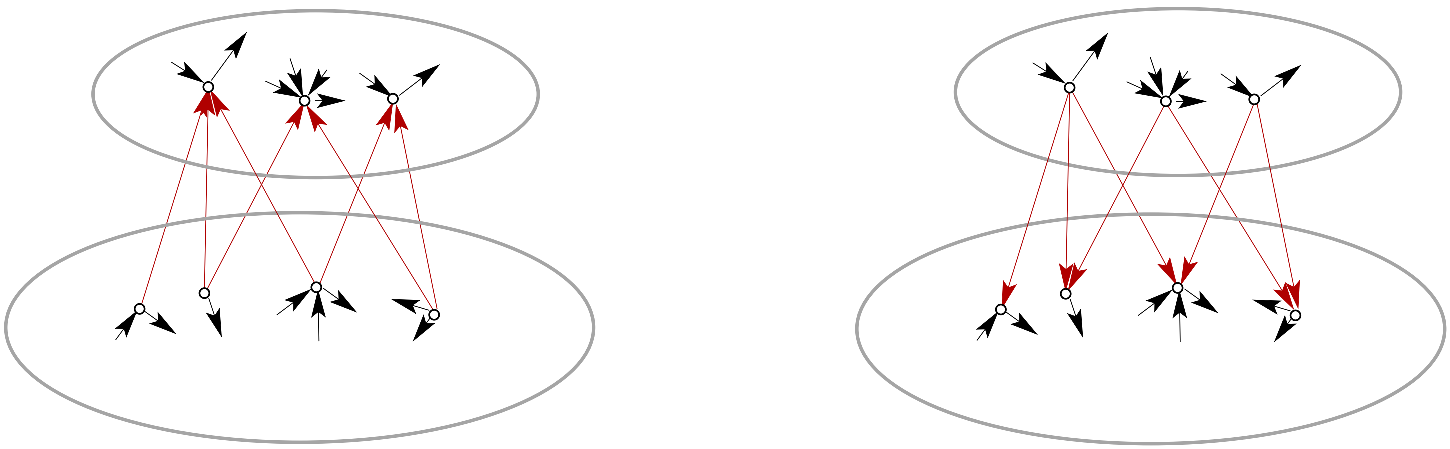






A Dual Construction

- Reorientations of directed cuts preserve flow-differences along cycles.

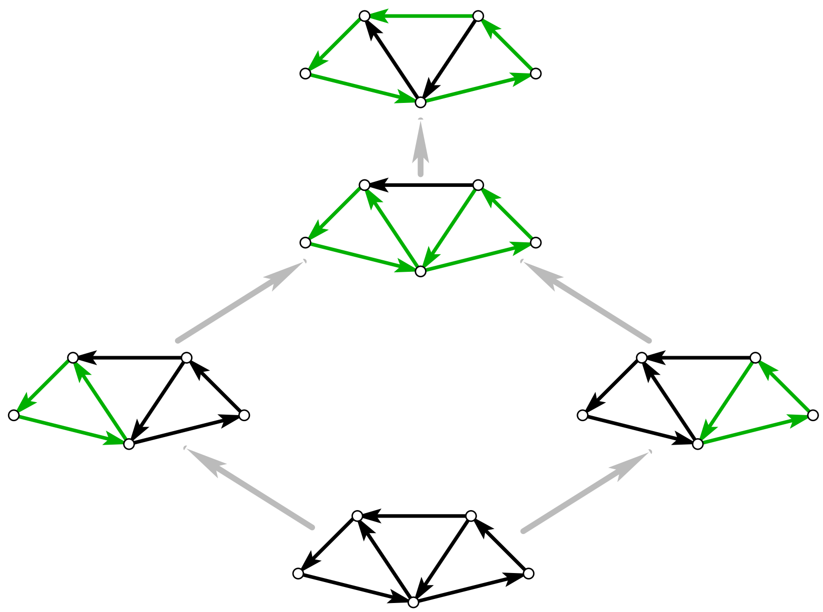


Theorem [Propp 1993]**.**
The set of all orientations of a graph G with prescribed flow-differences for all cycles has the structure of a distributive lattice.

Circulations in Planar Graphs

Theorem [Khuller, Naor and Klein 1993]**.**
The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.

$0 \leq f(e) \leq 1$



- Diagram edge $\sim$ add or subtract a unit of flow in ccw oriented facial cycle.

Δ-Bonds

$G = (V, E)$ a connected graph with a prescribed orientation.

With $x \in \mathbb{Z}^E$ and C cycle we define the circular flow difference

$$\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^{\mathcal{C}}$ and $\ell, u \in \mathbb{Z}^E$ let $\mathcal{B}_G(\Delta, \ell, u)$ be the set of $x \in \mathbb{Z}^E$ such that $\Delta_x = \Delta$ and $\ell \le x \le u$.

Theorem [Felsner, Knauer 2007]. $\mathcal{B}_G(\Delta, \ell, u)$ is a distributive lattice. The cover relation is vertex pushing.

Δ-Bonds as Generalization

$\mathcal{B}_G(\Delta, \ell, u)$ is the set of $x \in \mathbb{R}^E$ such that

- $\Delta_x = \Delta$ (circular flow difference)
- $\ell \leq x \leq u$ (capacity constraints).

Special cases:

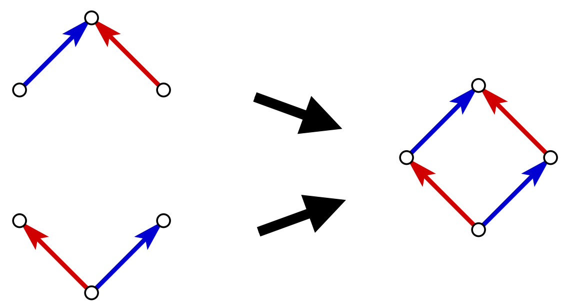
- c-orientations are $\mathcal{B}_G(\Delta, 0, 1)$ ($\Delta(C) = |C^+| - c(C)$).
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, u)$ (G^* the dual of G).
- α-orientations.

Diagrams of Distributive Lattices: A Characterization

A coloring of the edges of a digraph is a D-coloring iff

- arcs leaving a vertex have different colors.

- completion property:



Theorem.
A digraph D is connected, acyclic and admits a D-coloring
$\iff$ D is the diagram of a distributive lattice.

The End

Thank you.