Efficient Graph Packing via Game Coloring

H. A. Kierstead * A. V. Kostochka †

October 27, 2008

Abstract

The game coloring number $\mathrm{gcol}(G)$ of a graph G is the least k such that if two players take turns choosing the vertices of a graph then either of them can insure that every vertex has less than k neighbors chosen before it, regardless of what choices the other player makes. Clearly $\mathrm{gcol}(G) \leq \Delta(G) + 1$. Sauer and Spencer [20] proved that if two graphs G_1 and G_2 on n vertices satisfy $2\Delta(G_1)\Delta(G_2) < n$ then they pack, i.e., there is an embedding of G_1 into the complement of G_2. We improve this by showing that if $(\mathrm{gcol}(G_1) - 1)\Delta(G_2) + (\mathrm{gcol}(G_2) - 1)\Delta(G_1) < n$ then G_1 and G_2 pack. To our knowledge this is the first application of coloring games to a non-game problem.

1 Introduction

The purpose of this article is to demonstrate that game coloring can be a useful tool for attacking a non-game problem. We will use optimal strategies for Alice in the marking game to obtain an efficient packing of two graphs. It is our expectation that there will be other applications of this natural technique. A secondary goal is to illustrate a paradigm for generalizing results concerning graphs with bounded maximum degree.

The paper is organized as follows. In Sections 2, 3 and 4 we define and briefly discuss generalized coloring numbers, chromatic and marking games, and graph packing. In Section 5 we state and prove our main theorem. In Section 6 we make some concluding remarks.

The following graph Q_t will be used as an example several times. It is obtained from K_t by first duplicating edges so that every original edge has multiplicity t and then subdividing every edge exactly once. The t original vertices of K_t are called *branch vertices* of Q_t; the remaining $t\binom{t}{2}$ vertices are called *subdivision vertices*.

*Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287, USA. E-mail address: kierstead@asu.edu. Research of this author is supported in part by NSA grant H98230-08-1-0069

†Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA and Institute of Mathematics, Novosibirsk, Russia. E-mail address: kostochk@math.uiuc.edu. Research of this author is supported in part by NSF grant DMS-0650784 and by grant 06-01-00694 of the Russian Foundation for Basic Research.

2 Generalized coloring numbers

Fix a graph $G = (V, E)$ and denote the set of linear orderings on V by Π. Recall that the *coloring number* $\mathrm{col}(G)$ of G is the least integer k such that for some $L \in \Pi$ every vertex has less that k neighbors that precede it in L. It is well known that $\mathrm{col}(G) \leq d$ if and only if every subgraph has a vertex with degree less than d. This observation provides a polynomial time algorithm for determining $\mathrm{col}(G)$.

Starting with Chen and Schelp, several authors [6, 13, 11, 14] have considered useful variations of coloring number, with names such as arrangeability, admissibility and rank. Kierstead and Yang [16] standardized these ideas in terms of a hierarchy of coloring numbers as follows. Let $L \in \Pi$. A vertex z is *k-reachable* from a vertex x *with respect to* L iff $z <_L x$ and there exists an x, z-path P of length at most k such that $x <_L y$ for all interior vertices y of P. Let $R_{L,k}(x)$ denote the set of vertices that are k-reachable from x with respect to L. The *k-coloring number* was defined in [16] to be

$$\mathrm{col}_k(G) = 1 + \min_{L \in \Pi} \max_{x \in V} R_{L,k}(x).$$

The 1-coloring number of G is just the ordinary coloring number. The following easy inequalities hold for $j \leq k$:

$$\chi(G) \leq \mathrm{col}(G) \leq \mathrm{col}_j(G) \leq \mathrm{col}_k(G) \leq \Delta(G)(\Delta(G) - 1)^{k-1} + 1.$$

The graph Q_t satisfies $\mathrm{col}(Q_t) = 3$. Each of arrangeability, admissibility and rank can be polynomially bounded in terms of 2-coloring number and vice versa, but for specific applications each may have its own advantage. However, the 2-coloring number cannot be bounded in terms of the coloring number, since $\mathrm{col}_2(Q_t) = t+1$: Ordering all branch vertices before all subdivision vertices witnesses the upper bounds on both $\mathrm{col}_2(Q_t)$ and $\mathrm{col}(Q_t)$. For the lower bound on $\mathrm{col}_2(Q_t)$, note that for every ordering L, if x is the last branch vertex then $|R_{L,2}(x)| \geq t$ for every order L.

The 2-coloring number will have special importance for us. It, or slight variations, have been previously used in several different contexts. In a typical application one would like to show that a conclusion about the class of graphs with bounded maximum degree holds for the larger class of graphs with bounded coloring number. When we are unable to do this we may still be able to derive the conclusion for the intermediate class of graphs with bounded 2-coloring number. Here is the original example. Burr and Erdős [4] conjectured:

Conjecture 2.1 ([4])**.** *For every positive integer k there exists an integer C such that for every graph G on n vertices with $\mathrm{col}(G) \leq k$ and every graph H on Cn vertices, either $G \subseteq H$ or $G \subseteq \overline{H}$.*

Chvátal, Rödl, Szemerédi, and Trotter [7] proved a weaker version of the conjecture with $\Delta(G) \leq k$ instead of $\mathrm{col}(G) \leq k$. Then Chen and Schelp [6] introduced the original precursor of the 2-coloring number, and showed (essentially) that the conjecture holds with $\mathrm{col}_2(G) \leq k$. They demonstrated the significance of their result by showing (essentially) that the 2-coloring number of planar graphs is bounded, even though their maximum degree is not. Kierstead and Trotter [13] showed that it is bounded by 10, and gave an example to show that it is at least 8. A simpler example is the "buckyball", i.e, the dual of the graph formed by a 32-panel soccer ball (drawn with the stitching). To see this, fix an order L on the vertices and let x be the largest vertex that has a larger neighbor. After noting that if y and x have a common neighbor larger than x then $y \in R_{L,2}(x)$, it easily follows that $|R_{L,2}(x)| \geq 7$. Very recently Kierstead, Mohar, Špacapan, Yang and Zhu [12] proved the following theorem.

Theorem 2.2 ([12])**.** *Every planar graph G satisfies $\mathrm{col}_2(G) \leq 9$.*

A *vertex coloring* of G is degenerate if the union U of any k color classes satisfies $\mathrm{col}(G[U]) \leq k$. In particular, setting $k = 1$ we see that a degenerate coloring is a proper coloring; moreover setting $k = 2$ we see that it is an acyclic coloring. The degenerate chromatic number $\chi_d(G)$ of G is the least k such that G has a degenerate coloring with k colors. While $\mathrm{col}(Q_t) = 3$, we have $\chi_d(G) \geq \chi_a(Q_t) = t$: If we try to use fewer colors then two branch vertices will have the same color, and two of their common neighbors will also share a color, creating a bicolored 4-cycle. However $\chi_d(G) \leq \Delta(G)(\Delta(G)-1)+1$: Fix an order L, and color sequentially so that each new vertex x gets a color that has has not been used on $|R_{L,2}(x)|$. Then each vertex y will have at most one neighbor that precedes it in each color class. This idea leads to the following observation [12], as well as its list coloring counterpart.

Theorem 2.3. *Every graph G satisfies $\chi_d(G) \leq \mathrm{col}_2(G)$. In particular $\chi_d(G) \leq 9$, if G is planar.*

Finally, we remark that generalized coloring numbers are related to the concept of bounded expansion classes (see Nešetřil and Ossona de Mendez [18]). Zhu [24] has very recently proved that a graph class $\mathcal{K}$ has bounded expansion if and only if there exists a function f such that every graph $G \in \mathcal{K}$ satisfies $\mathrm{col}_k(G) \leq f(k)$ for all natural numbers k.

3 Games

The *chromatic game* is played on G by two players, Alice and Bob, with a fixed set of colors C. The game begins with Bob deciding which player will make the first move. Once this decision is made, the players take turns coloring the vertices with colors from C with *legal* colors, i.e., so that adjacent vertices receive distinct colors. Alice wins if the graph is eventually properly colored; Bob wins if at some time there is an uncolored vertex that cannot be legally colored. The game chromatic number $\chi_g(G)$ of G is the least integer k such that Alice has a winning strategy when $|C| = k$; for a class $\mathcal{C}$ of graphs $\chi_g(\mathcal{C}) = \max_{G \in \mathcal{C}} \chi(G)$. This game was originally introduced by Brams and published by Gardner [10] for planar graphs. It was reinvented and introduced to the mathematical community by Bodlaender [1]. Faigle, Kern, Kierstead and Trotter [9] showed that the game chromatic number of the class of forests is 4. Kierstead and Trotter [13] used a variant of the 2-coloring number to show that the game chromatic number of planar graphs is at most 33. This result was improved by a series of papers [8, 21, 11, 23]. The following bound obtained by Zhu is currently the best.

Theorem 3.1 ([23]). *Every planar graph G satisfies $\chi_g(G) \leq 17$.*

Proofs in [9, 21, 11, 23] are all obtained by bounding a new game parameter (implicitly in [9]), the *game coloring number*. The *marking game* is played by Alice and Bob on G. Bob decides who goes first and then the players take turns choosing unchosen vertices until all vertices have been chosen. Let L be the order in which the vertices are chosen, where the first vertex is smallest. The score of the game is the maximum k such that in L some vertex has $k-1$ smaller neighbors. Alice's goal is to keep the score low, while Bob's goal is to force it high. The *game coloring number* $\mathrm{gcol}(G)$ is the least k such that Alice can always obtain a score of at most k. The game k-coloring number $\mathrm{gcol}_k(G)$ of G is defined analogously. Clearly $\mathrm{gcol}(G) = \Delta(G) + 1$ if G is regular. Moreover, if Alice interprets Bob's moves in the chromatic game as choices of vertices in the marking game, then she can choose vertices to color so that no vertex has $\mathrm{gcol}(G)$ colored neighbors. It follows that:

$$\chi(G) \leq \mathrm{col}(G) \leq \mathrm{gcol(G)} \leq \Delta(G) + 1 \text{ and } \chi_g(G) \leq \mathrm{gcol}(G).$$

However $\mathrm{gcol}(G)$ cannot be bounded in terms of $\mathrm{col}(G)$, since $\mathrm{col}(Q_t) = 3$, but $\mathrm{gcol}(Q_t) > s = \log_2(t)$ for $t = 2^s$: Argue by induction on s and observe that the base step $s = 0$ is trivial. First Bob chooses $\frac{1}{2}t$ subdivision vertices that dominate all branch vertices. Alice can respond by choosing at most $\frac{1}{2}t$ branch vertices, and so there exists a set U of $\frac{1}{2}t$ unchosen branch vertices, each of which is adjacent to a chosen subdivision vertex. Now apply the induction hypothesis to $G[U]$. The following theorem is the third example of our paradigm.

Theorem 3.2 ([11]). *Every graph G satisfies* $\mathrm{gcol}(G) \leq 3\mathrm{col}_2(G) - 1$.

Kierstead and Trotter [14] showed that if G is planar then $\mathrm{col}_4(G)$ is bounded. They used this to show that $\mathrm{gcol}_2(G)$ is bounded, which was key in showing that the *game oriented chromatic number* of G is bounded. Kierstead and Yang [16] showed that the *game k-coloring number* is bounded in terms of the $2k$-coloring number and that the k-coloring number of planar graphs is bounded.

4 Packing

Two n-vertex graphs G_1 and G_2 *pack*, if there exists an edge-disjoint placement of these graphs onto the same set of n vertices. By definition, G_1 and G_2 pack, if G_1 is a subgraph of the complement $\overline{G}_2$ of G_2. A number of basic graph-theoretic problems can be expressed in a unified and symmetric form as graph packing problems. Among such problems are Turán-type, Ramsey-type, coloring and equitable coloring problems. Since even partial cases of graph packing problems are NP-hard, study of extremal problems was initiated in 70s by seminal papers of Bollobás and Eldridge [4], Sauer and Spencer [20], and Catlin [5]. Their conditions involved maximum degrees of graphs. For example, Sauer and Spencer [20] proved the following result.

Theorem 4.1 ([20]). *If G_1 and G_2 are graphs on n vertices and $2\Delta(G_1)\Delta(G_2) < n$, then G_1 packs with G_2.*

There were several refinements of Theorem 4.1. One of them involves the *maximum edge degree* $\theta(G) := \max_{xy\in E(G)}(d(x)+d(y))$. Clearly, $\theta(G)$ is closely related to the maximum degree of the line graph $L(G)$:

$$\theta(G) = \Delta(L(G)) + 2,$$

and $\theta(G) \leq 2\Delta(G)$ for every graph G. Kostochka and Yu showed that $2\Delta(G_1)$ could be replaced by $\theta(G_1)$ in Theorem 4.1, and moreover:

Theorem 4.2 ([17]). *If two n-vertex graphs G_1 and G_2 satisfy the inequality $\theta(G_1)\Delta(G_2) \leq n$, then G_1 and G_2 pack, with the following exceptions:*
(I) G_1 is a perfect matching and G_2 either is $K_{n/2,n/2}$ with $n/2$ odd or contains $K_{n/2+1}$;
(II) G_2 is a perfect matching, and G_1 either is $K_{r,n-r}$ with r odd or contains $K_{n/2+1}$.

Bollobás, Kostochka and Nakprasit showed that if one of the two graphs has small coloring number then much weaker conditions on $\Delta(G_1)$ and $\Delta(G_2)$ imply the existence of a packing.

Theorem 4.3 ([3]). *Let $\mathrm{col}(G_1) \geq 3$. If G_1 and G_2 are graphs on n vertices with*

$$40\Delta(G_1)\ln\Delta(G_2) < n \text{ and } 40\mathrm{col}(G_1)\Delta(G_2) < n,$$

then G_1 packs with G_2.

Let G_1 and G_2 be graphs on n vertices. We could write the hypothesis of Theorem 4.1 as a two term sum. Then motivated by Theorem 4.3 and our paradigm, we might consider strengthening Theorem 4.1 to

$$\exists C\ C\mathrm{col}(G_1)\Delta(G_2) + C\mathrm{col}(G_2)\Delta(G_1) < n \text{ implies } G_1 \text{ packs with } G_2.$$

This is too rash to call a conjecture, and we certainly cannot prove it, but the following weakening according to our paradigm is a corollary to our main result, Theorem 5.1.

Corollary 4.4. *Let G_1 and G_2 be graphs on n vertices. If*

$$3\mathrm{col}_2(G_1)\Delta(G_2) + 3\mathrm{col}_2(G_2)\Delta(G_1) < n$$

then G_1 packs with G_2.

5 The main result

Our main result, Theorem 5.1, proved in this section, not only is a natural example of our paradigm, but also to our knowledge, is the first use of game coloring to prove a non-game result. The statement of the theorem is stronger than that of the Sauer-Spencer Theorem, since $\mathrm{gcol}(G) - 1 \leq \Delta(G)$ for every graph G. The proof is motivated by the classical back-and-forth construction, showing that any two countable dense linear orders without endpoints are isomorphic.

Theorem 5.1. *If G_1 and G_2 are n-vertex graphs and*

$$(\mathrm{gcol}(G_1)-1)\Delta(G_2)+(\mathrm{gcol}(G_2)-1)\Delta(G_1)<n, \tag{5.1}$$

then G_1 and G_2 pack.

Proof. For $i \in [2]$ set $G_i := (V, E_i)$, $d_i(v) := d_{G_i}(v)$, $g_i := \mathrm{gcol}(G_i) - 1$ and $\Delta_i = \Delta(G_i)$. A packing of G_1 with G_2 will be viewed as a bijection $f : V \to V$ such that every $uv \in E_1$ satisfies $f(u)f(v) \notin E_2$.

Our plan is to construct a packing of G in stages. At the end of stage s we will have a partial packing f^s of $H_1^s \subseteq G_1$ onto $H_2^s \subseteq G_2$, where $|H_i^s| = s$. We will maintain the invariant $H_i^s \subseteq H_i^t$, but not necessarily $f^s \subseteq f^t$, for $s < t$. At stage s we will choose two new vertices x_1^s and x_2^s with $x_i^s \in V(G_i - H_i)$. These choices will be interpreted as moves in two coloring games, Game$_1$ played on G_1 and Game$_2$ played on G_2. The players for Game$_i$ are Alice$_i$ and Bob$_i$, although as we shall see it would perhaps be more informative to change the names to Algorithm and Black Box. We assume that each Alice$_i$ plays with an optimal strategy witnessing the game coloring number of G_i. We will provide each Bob$_i$ with a strategy designed to produce a packing.

Recall that Bob$_i$ decides who plays first. For our purposes, we have Bob$_i$ choose to play first if and only if $i = 2$. Let $x_i^1, x_i^2, \cdots, x_i^n$ be the sequence of moves in Game$_i$; so Bob$_i$ plays x_i^s if and only if $i + s$ is odd. The optimal strategy of Alice$_i$ must be able to handle this situation. At stage 1 of the construction Alice$_1$ chooses x_1^1 in accordance with her optimal strategy. Our construction will generate $f^1(x_1^1) = x_2^1$, which is interpreted as Bob$_2$'s first play in Game$_2$. At stage 2, Alice$_2$ responds to Bob$_2$ by choosing x_2^2 in accordance with her optimal strategy. Then the construction will generate $(f^2)^{-1}(x_2^2) = x_1^2$, which is interpreted as Bob$_1$'s response to x_1^1. Continuing in this back-and-forth fashion, we will generate the two sequences $x_i^1, x_i^2, \cdots, x_i^n$, $i \in [2]$. By the definition of game coloring number, Alice$_i$'s optimal strategy insures that the set $N_i^+(w)$ of neighbors of w preceding w in H_i^s has cardinality at most g_i. It remains to show how the construction will generate x_i^s for $i + s$ odd.

Argue by induction on s. The base step $s = 0$ is trivial, so consider the induction step $s + 1$. By the induction hypothesis, f^s is a packing of H_1^s with H_2^s. Assume $s + 1$ is odd as the even case is symmetrical. So Alice$_1$ has just chosen $x := x_1^{s+1}$. For $x' \in V(G_2 - H_2^s)$ define the set $C(x')$ by

$$C(x') := \{y \in H_1^s : xy \in E_1 \text{ and } x'f^s(y) \in E_2\}.$$

Case 1: $C(x') = \emptyset$ for some $x' \in V(G_2 - H_2^s)$. Setting $x_2^{s+1} := x'$ and $f^{s+1} = f^s \cup \{(x, x')\}$, we are done.
Case 2: $C(x') \neq \emptyset$ for all $x' \in V(G_2 - H_2^s)$. Fix $x' \in V(G_2 - H_2^s)$; set $x_2^{s+1} := x'$ and $C := C(x')$. We will show that for some *good* $z \in V(H_1^{s+1})\backslash C$, the mapping $\varphi := \varphi_z$ obtained from $f := f^s \cup \{(x, x')\}$ by switching the images of x and z is a packing of H_1^{s+1} with H_2^{s+1}. Suppose $z \in V(H_1^{s+1})\backslash C$ is *bad*, i.e., not good. Then there exists an edge $uv \in E_1$ with $\varphi(u)\varphi(v) \in E_2$. Since f^s packs H_1^s with H_2^s, at least one, say v, of u, v is in $\{x, z\}$. Since $z \notin C$, we have $u \notin \{x, z\}$.

Suppose $v = x$. Then $xu \in E_1$ and $\varphi(x)\varphi(u) = f^s(z)f^s(u) \in E_2$. Thus $f^s(z) \in P$, where

$$P := \bigcup_{u \in N_1^+(x)} N_2(f^s(u)).$$

Since $|N_1^+(x)| \leq g_1$ and $|N_2(w)| \leq \Delta_2$ for all $w \in V$, we have $|P| \leq g_1\Delta_2$. But this is an overcount, since we count $x' \in N_2(f^s(u))$ for each $u \in C$. With this observation we obtain

$$|P| \leq g_1\Delta_2 - |C| + 1.$$

Moreover, by the case, $V(G - H_2^s) \subseteq P$.

Suppose $v = z$. Then $zu \in E_1$ and $\varphi(z)\varphi(u) = x'f^s(u) \in E_2$. Then $z \in Q$, where

$$Q := \bigcup_{u' \in N_2^+(x')} N_1((f^s)^{-1}(u')).$$

Since $N_2^+(x') \leq g_2$ and $|N_1((f^s)^{-1}(u')| \leq \Delta_1$ for all $u' \in V$, we have $|Q| \leq g_2\Delta_1$. Again, this is an overcount, since we count x for each $u' \in f^s(C)$. Thus

$$|Q| \leq g_2\Delta_1 - |C| + 1.$$

Note that $x' \in P$ and $x \in Q$, since $C \neq \emptyset$. Thus

$$|P \cup f(Q)| \leq g_1\Delta_2 + g_2\Delta_1 - 2|C| + 2 - 1 < n - |C|.$$

It follows that there exists $z' \in V - (P \cup f(Q) \cup f(C))$. By the case, $z' \in V(H_2^s)$. So $z := f^{-1}(z')$ is good, and thus $f^{s+1} := \varphi_z$ is a packing of H_1^{s+1} with H_2^{s+1}. This completes the induction step and the proof. □

6 Conclusion

Recall from Section 2 that there is a polynomial time algorithm for determining $\mathrm{col}(G)$. Given the many applications of the 2-colorinng number, it is natural to ask:

Problem 1. *Is there a polynomial time algorithm for calculating* $\mathrm{col}_2(G)$*?*

In [14] it is shown that $\mathrm{gcol}(G)$, and thus also $\mathrm{col}_2(G)$, cannot be bounded in terms of $\chi_a(G)$. A positive answer to the next problem would provide another application of game coloring number to a non-game problem.

Problem 2. *Can* $\mathrm{col}_2(G)$ *and/or* $\chi_a(G)$ *be bounded in terms of* $\mathrm{gcol}(G)$*?*

In light of Theorem 4.2 it would be interesting to find a good bound on $\mathrm{gcol}(G)$ in terms of $\theta(G)$. Here is a start. If G is regular then $\mathrm{gcol}(G) = \frac{1}{2}\theta(G) + 1$. So suppose G is not regular. The *high* degree vertices v satisfying $d(v) > \frac{1}{2}\theta(G)$ form an independent set. Only these vertices can be adjacent to more than $\frac{1}{2}\theta(G)$ previously chosen vertices. So Alice will always choose high degree vertices, if possible. Similarly, Bob will only choose *low* degree vertices v satisfying $d(v) \leq \frac{1}{2}\theta(G)$ which threaten unchosen high degree vertices. We may assume that the low degree vertices also form an independent set.

Proposition 6.1. *For every positive integer $k \geq 3$ there exists a graph G with*

$$k+1 = \Delta(G)+1 = \operatorname{gcol}(G) = \frac{\theta(G)+3}{2}.$$

Proof. Fix k. Let G be an X, Y-bigraph, where

$$X := \{x_{i,j} : i \in [k-1], j \in [k]\} \text{ and } Y := \{y_j : j \in [k]\} \cup \{z_{i,j} : i \in [k-1], j \in [k]\}.$$

Define $E := E(G)$ by

$$E := \{y_j x_{i,j} : i \in [k-1], j \in [k]\} \cup (z_{i,j} x_{i,h} : i \in [k-1], j, h \in [k], j \neq h\}.$$

Then every vertex in X has degree k and every vertex in Y has degree $k-1$. So $\operatorname{gcol}(G) \leq k+1$ and $\theta(G) = 2k-1$. Thus it suffices to show that Bob has a strategy for obtaining a score of at least $k+1$.

While possible, Bob will always choose vertices from Y, and as above, we may assume that Alice always tries to choose a vertex from X. Bob starts and begins by choosing $y_1, \dots, y_k$. After Alice's first k moves, there exists $i \in [k-1]$ such that at most one vertex of $X_i := x_{i,1}, \dots, x_{i,k}$ has been chosen. Without loss of generality, no vertex of X_i, other than possibly $x_{i,k}$, has been chosen. Bob chooses $z_{i,k}$. From now on, whenever Alice chooses a vertex $x_{i,j}$ Bob chooses a vertex $z_{i,h}$ with $h = j$ if possible. When Alice chooses the last vertex of X_i, say $x_{i,n}$, each of its neighbors in the set $\{z_{i,j} : j \in [k]\} + y_j - z_{i,n}$ will already be chosen. □

Proposition 6.2. *Suppose that G is an X, Y-bigraph such that*

$$d(y) = d \leq D = d(x) \textit{ for all } y \in Y, x \in X.$$

Then $\operatorname{gcol}(G) \leq D + 2 - \lfloor \frac{D}{d} \rfloor$.

Proof. Let $k = \lfloor \frac{D}{d} \rfloor$. Obtain an X', Y-bigraph G' from G by splitting each vertex $x \in X$ into k vertices $x_1, \dots, x_k$ of degree d or $d+1$ so that $\{N(x_i) : i \in [k]\}$ partitions $N(x)$. By Hall's Theorem, G' has a matching that saturates X'. Recombining the vertices $x_1, \dots, x_k$ into x yields a set $\{S_x : x \in X\}$ of disjoint stars, where S_x has center x and k edges.

Whenever Bob chooses a vertex y in a star S_x Alice responds by choosing x if it has not already been chosen. This ensures that when x is finally chosen it will have at most $D-k+1$ previously chosen neighbors. □

Problem 3. *Does every graph G satisfy* $\operatorname{gcol}(\mathrm{G}) \leq \frac{\theta(\mathrm{G})+3}{2}$*?*

Dedication. Professor Trotter has had a profound influence on our careers as well as the careers of many other mathematicians. It is a pleasure to honor him on the occasion of his 65th birthday and wish him many happy returns of the day.

References

[1] Bodlaender, H. L.: On the complexity of some coloring games, Internat. J. Found. Comput. Sci. 2 (1991) 133–147.

[2] Bollobás, B. and Eldridge, S. E.: Packing of graphs and applications to computational complexity, J. Comb.Theory Ser. B, 25 (1978), 105–124.

[3] Bollobás, B., Kostochka, A. V. and Nakprasit, K.: On two conjectures on packing of graphs, Combinatorics, Probability and Computing 14 (2005), 723–736.

[4] Burr, S.A. and Erdős, P.: On the magnitude of generalized ramsey numbers, Infinite and Finite Sets, Hajnal, A., Rado, R. and Sós, V. T., Colloq. Math. Soc. Janos Bolyai, vol. 1, North Holland, Amsterdam / London (1975).

[5] Catlin, P. A.: Subgraphs of graphs. I. Discrete Math., 10 (1974), 225–233.

[6] Chen, G. and Schelp, R. H.: Graphs with linearly bounded Ramsey numbers, J. Combin. Theory Ser. B 57 (1993), 138–149.

[7] Chvatál, V., Rodl, V., Szemerédi, E. and Trotter, W. T.: The ramsey number of a graph of bounded degree, J. Combin. Theory (B) 34 (1983) 239–243.

[8] Dinski, T. and Zhu, X.: A bound for the game chromatic number of graphs, Discrete Math. 196 (1999), 109–115.

[9] Faigle, U., Kern, W., Kierstead, H. A. and Trotter, W. T.: On the game chromatic number of some classes of graphs, Ars Combin. 35 (1993), 143–150.

[10] Gardner, M.: Mathematical games, Scientific American (April, 1981) 23.

[11] Kierstead, H. A.: A simple competitive graph coloring algorithm, J. Combin. Theory (B) 78 (2000), 57–68.

[12] Kierstead, H. A., Mohar, B., Špacapan, S., Yang, D. and Zhu, X.: The two-coloring number and degenerate colorings of planar graphs, submitted.

[13] Kierstead, H. A. and Trotter, W. T.: Planar graph coloring with an uncooperative partner, J. Graph Theory 18 (1994), 569–584.

[14] Kierstead, H. A. and Trotter, W. T.: Competitive colorings of oriented graphs. In honor of Aviezri Fraenkel on the occasion of his 70th birthday, Electron. J. Combin. 8 (2001), Research Paper 12, 15 pp. (electronic).

[15] Kierstead, H. A. and Tuza, Zs.: Marking games and the oriented game chromatic number of partial k-trees, Graphs and Combinatorics 19 (2003), 121–129.

[16] Kierstead, H. A. and Yang, D.: Orderings on graphs and game coloring number, Order 20 (2003), 255–264 (2004).

[17] Kostochka, A. V. and Yu, G.: An Ore-type analogue of the Sauer-Spencer Theorem, Graphs and Combinatorics 23 (2007), 419–424.

[18] Nešetřil, J. and Ossona de Mendez, P.: Grad and classes with bounded expansion I. Decompositions. KAM-DIMATIA Series, no. 2005-739.

[19] Nešetřil, J. and Sopena, E.: On the oriented game chromatic number. In honor of Aviezri Fraenkel on the occasion of his 70th birthday, Electron. J. Combin. 8 (2001), Research Paper 14, 13 pp. (electronic).

[20] Sauer, N. and Spencer, J.: Edge disjoint placement of graphs, J. Combin. Theory Ser. B 25 (1978) 295–302.

[21] Zhu, X.: Game coloring number of planar graphs, J. Combin. Theory (B) 75 (1999), 245–258.

[22] Zhu, X.: The game coloring number of pseudo partial k-trees, Discrete Math. 215 (2000), 245–262.

[23] Zhu, X.: Refined activation strategy for the marking game, J. Combin. Theory (B) 98 (2008) 1–18.

[24] Zhu, X.: Colouring graphs with bounded generalized colouring number, Discrete Math., to appear.