Arbitrarily vertex decomposable graphs

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AVD graphs

Definition.
Let $G = (V, E)$ be a graph of order n and let $\mathcal{P}$ be a graph property. A sequence $(n_1, ..., n_k)$ of non-negative integers is called *admissible for G (with respect to $\mathcal{P}$)* if

- for each its element n_i there exists an induced subgraph of G of order n_i having property $\mathcal{P}$ and
- $\sum_i n_i = n$.

AVD graphs

An admissible sequence (n_i) is *realizable* in G if there exists a partition $V_1, \ldots, V_k$ of the vertex set V of G such that

- $|V_i| = n_i$
- the induced subgraphs $G[V_i]$ have property $\mathcal{P}$.

A graph G is said to be *arbitrarily vertex decomposable (with respect to $\mathcal{P}$)* (AVD for short) if each admissible sequence is realizable.
If k is fixed we speak about k-AVD graphs.

Other properties

There are results concerning the properties:

- *to be hamiltonian* ($n_i \geq 3$)
- *to be without isolated vertices* ($n_i \geq 2$)

Property $\mathcal{P}$: *to be hamiltonian*

Theorem(M.Aigner and S.Brandt, 1993) If $\delta(G) \geq \frac{2n-1}{3}$ then G contains each graph H with $\Delta(H) \leq 2$.
In particular, for $\Delta(H) = 2$, we have
Theorem
If $\delta(G) \geq \frac{2n-1}{3}$ then G is AVD (with respect to $\mathcal{P}$).

$\mathcal{P}$: *to be without isolated vertices;* k *- fixed*

In 1975 A.Frank stated the following conjecture.
Conjecture If G is connected and $\delta(G) \geq k$, then G is k-AVD.

Still open. Satisfied for

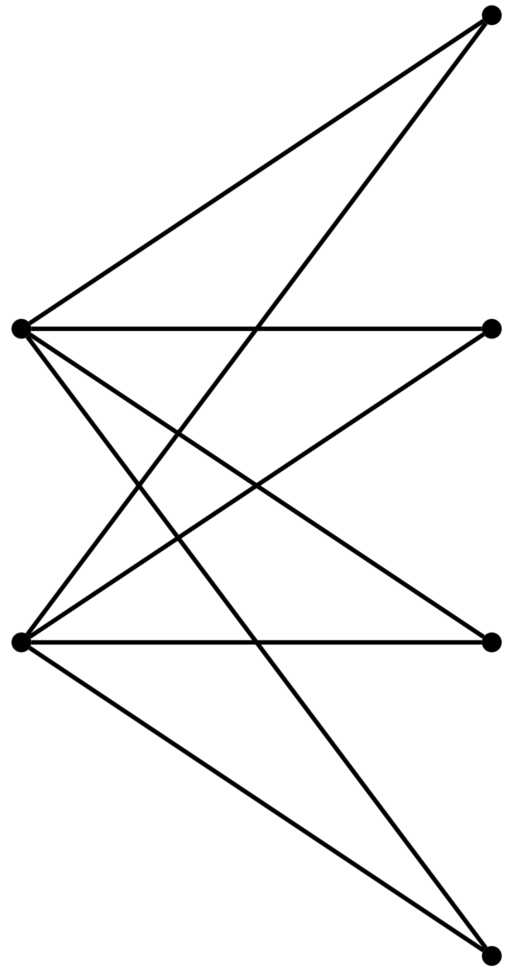
- $k = 2$ (Maurer, 1979)
- $k = 3$ (Linial, 1984)
- $n_i = 2$ for $1 \leq i \leq k - 1$ (Linial)
- $2 \leq n_i \leq 3$ dla $1 \leq i \leq k$ (Enomoto, A.Kaneko and Zs.Tuza, 1987)
- (H.Enomoto, S. Matsunaga and K. Ota, 1996)

Property $\mathcal{P}$: *to be connected*; k - fixed

L.Lovász (1977) and E.Győri (1978) proved that:

Theorem

k-connected $\Longrightarrow$ k-AVD.



6=2+2+2

Figure 1: $K_{2,4}$

Examples of AVD trees

- Paths

Examples of AVD trees

- Paths
- Caterpillars with one leg $\mathrm{Cat}(a, b)$, if a and b are coprime

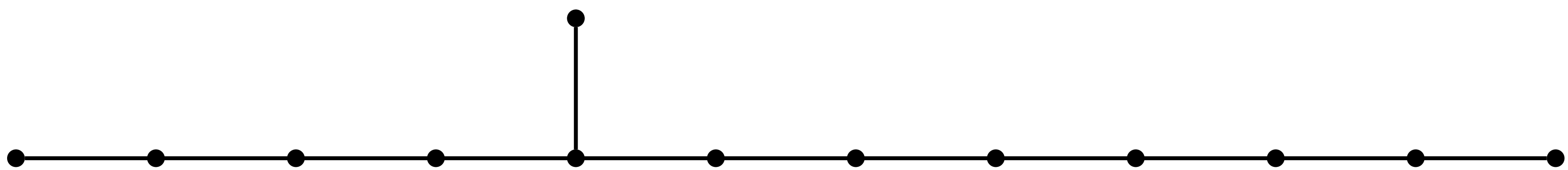


Figure 3: $\mathrm{Cat}(5, 8)$

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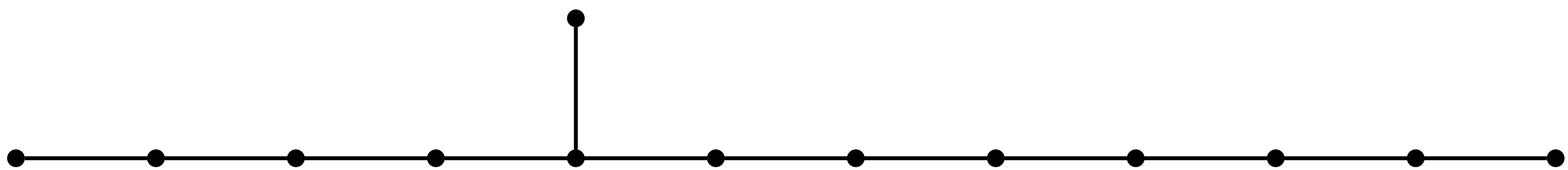

Figure 4: $\mathrm{Cat}(5,8)$

- Some other caterpillars with two or three legs.

Theorem (S.Cichacz, A. Görlich, A.Marczyk, J.Przybyło, MW)
Let $T=(V,E)$ be a caterpillar of order n with two single legs attached at x and y. Then T is avd if and only if the following holds:

1^0 $(l_x(T), r_x(T)) = 1$;

2^0 $(l_y(T), r_y(T)) = 1$;

3^0 $(l_x(T), r_y(T)) = 1$;

4^0 $(l_y(T), r_x(T)) < l_y - l_x$ or $n \equiv 1$ (mod $(l_y(T), r_x(T)))$;

5^0 $n \neq \alpha l_x(T) + \beta l_y(T)$ for any $\alpha, \beta \in \mathbf{N}$;

6^0 $n \neq \alpha r_x(T) + \beta r_y(T)$ for any $\alpha, \beta \in \mathbf{N}$.

A general result on AVD trees

- D.Barth, O.Baudon and J.Puech, Decomposable trees: a polynomial algorithm for tripodes, *Discrete Appl. Math.* **119** (2002), 205–216.

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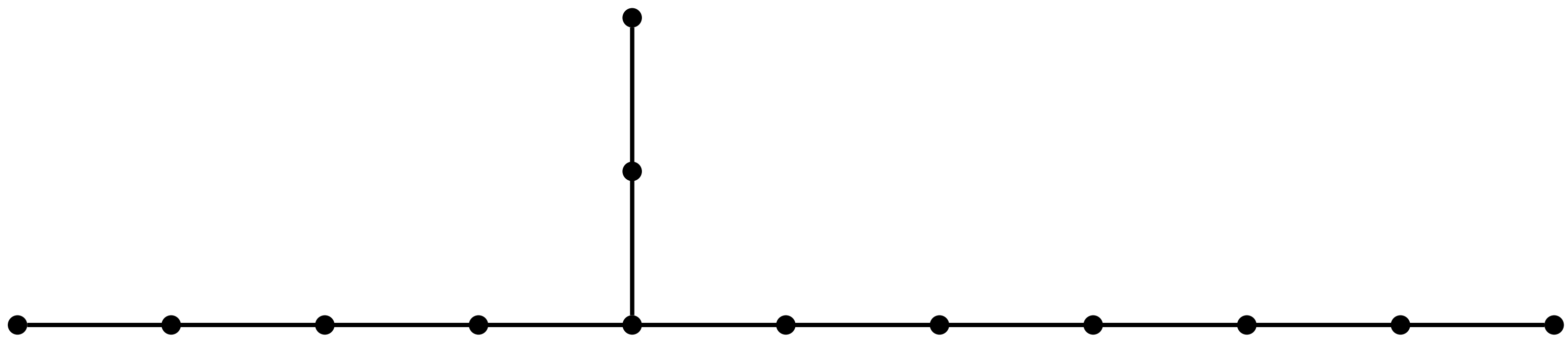
- D. Barth and H. Fournier, A Degree Bound on Decomposable Trees, *Discrete Math.* **306** (2006), 469–477.

Main result on trees

Theorem (D. Barth and H. Fournier)
If $\Delta(T) \geq 5$ then the tree T is not AVD.

Some questions: tripodes = 3-spiders



- Tripode $S(a_1, a_2, a_3)$; $a_1 \leq a_2 \leq a_3$.

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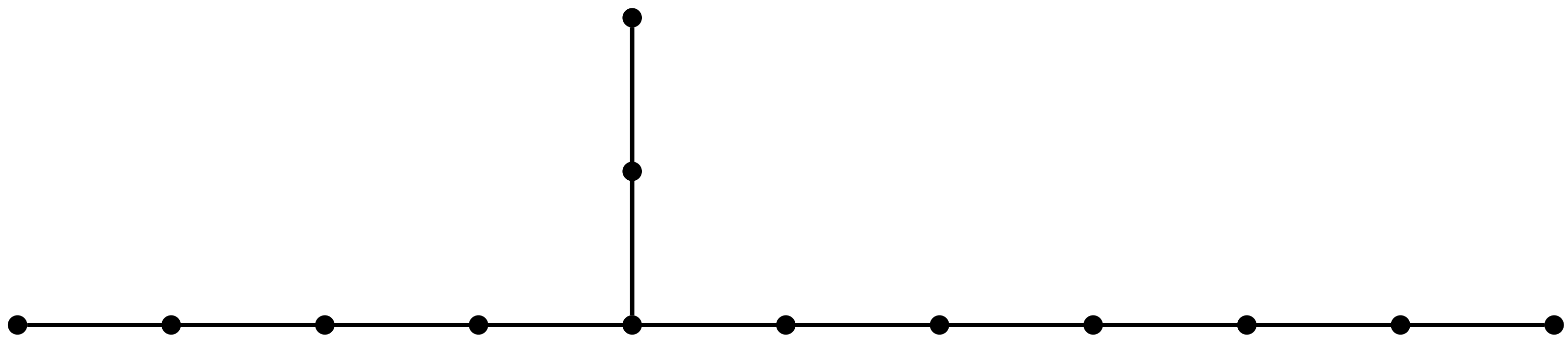

- Tripode $S(a_1, a_2, a_3)$; $a_1 \leq a_2 \leq a_3$.
- In our example: $a_1 = 3$, $a_2 = 5$, $a_3 = 7$, $n = 13$.

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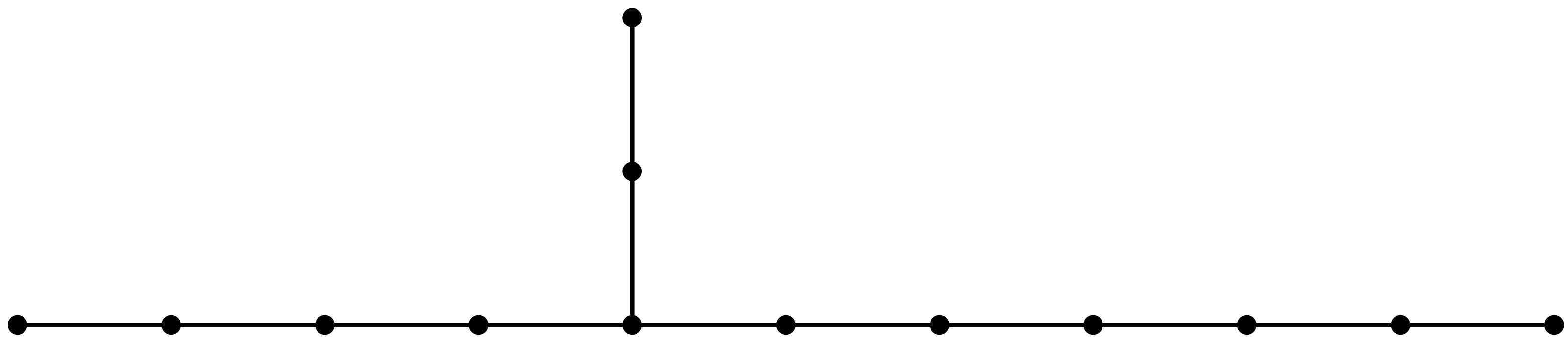

- Tripode $S(a_1, a_2, a_3)$; $a_1 \leq a_2 \leq a_3$.
- In our example: $a_1 = 3$, $a_2 = 5$, $a_3 = 7$, $n = 13$.
- Question:. Can a_1 be arbitrarily large? (There are AVD tripodes with $a_1 = 20$)

Some questions: 4-spiders

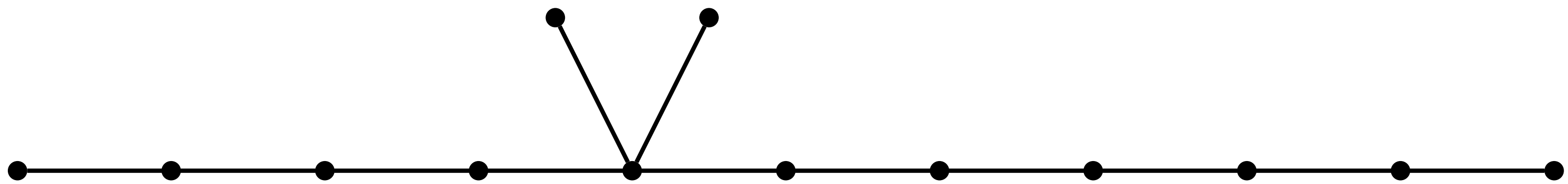

Figure 5: $S(2, 2, 5, 7)$

- 4-spider $S(a_1, a_2, a_3, a_4)$; $a_1 \leq a_2 \leq a_3 \leq a_4$.

Some questions: 4-spiders

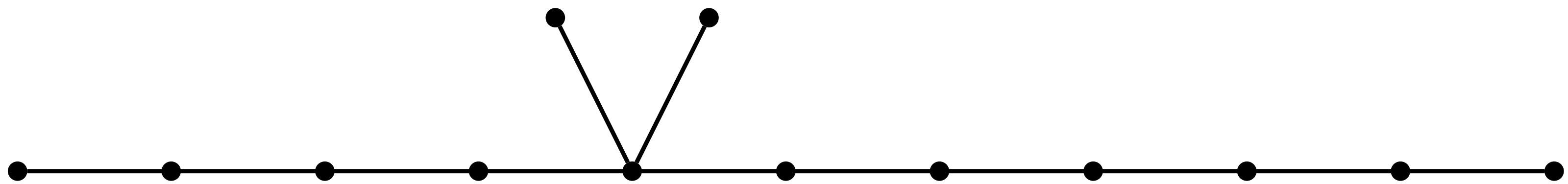

Figure 6: $S(2, 2, 5, 7)$

- 4-spider $S(a_1, a_2, a_3, a_4)$; $a_1 \leq a_2 \leq a_3 \leq a_4$.
- In our example: $a_1 = 2$, $a_2 = 2$, $a_3 = 5$, $a_4 = 7$.

Some questions: 4-spiders

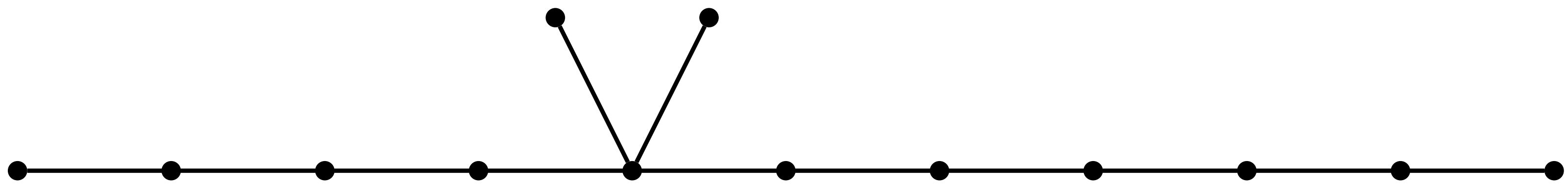

Figure 7: $S(2, 2, 5, 7)$

- 4-spider $S(a_1, a_2, a_3, a_4)$; $a_1 \leq a_2 \leq a_3 \leq a_4$.
- In our example: $a_1 = 2$, $a_2 = 2$, $a_3 = 5$, $a_4 = 7$.
- Theorem (D. Barth and H. Fournier)
  If a tree T is AVD, then each vertex of T of degree four is adjacent to a leaf.

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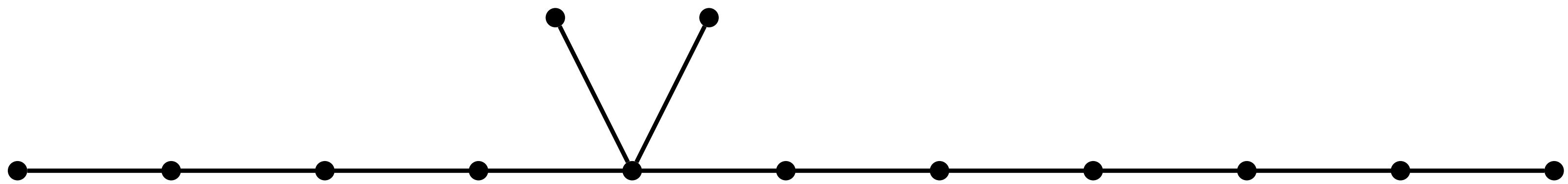
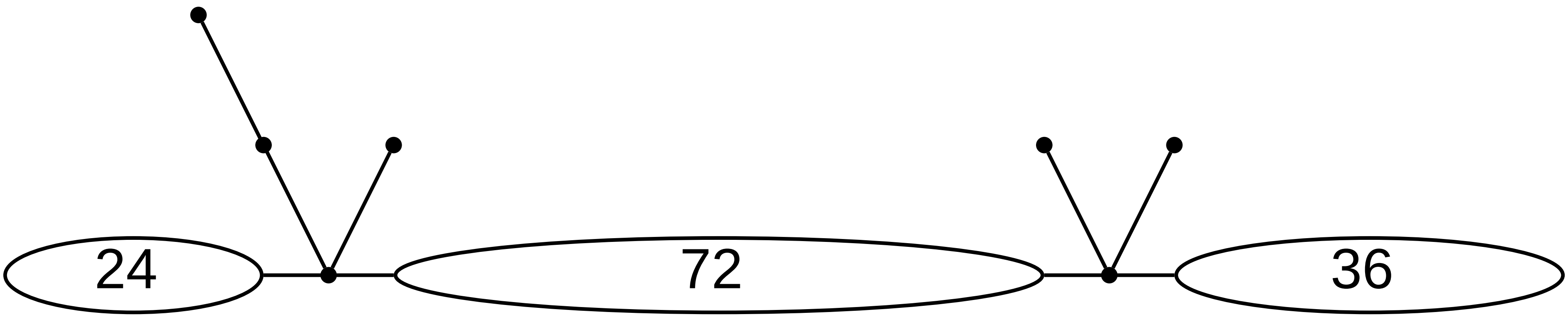


Figure 8: $S(2, 2, 5, 7)$

- 4-spider $S(a_1, a_2, a_3, a_4)$; $a_1 \leq a_2 \leq a_3 \leq a_4$.
- In our example: $a_1 = 2$, $a_2 = 2$, $a_3 = 5$, $a_4 = 7$.
- Theorem (D. Barth and H. Fournier)
  If a tree T is AVD, then each vertex of T of degree four is adjacent to a leaf.
- Question:. Can a_2 be arbitrarily large? (There are AVD 4-spiders with $a_2 = 3$)

AVD trees with two vertices of degree four

We know only one example of such a tree.



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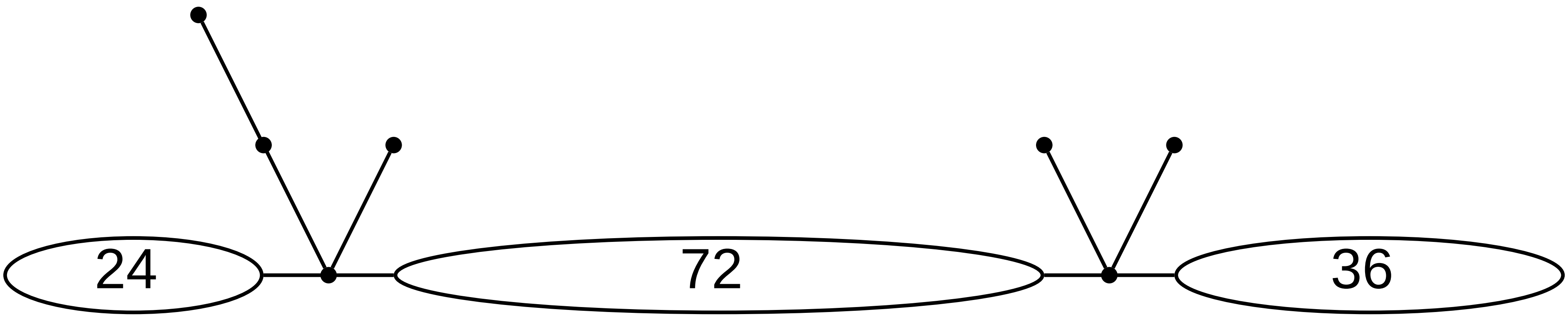

- Questions:
  Can an AVD tree have three vertices of degree four?
  Are there any other AVD trees with two vertices of degree four?

Suns

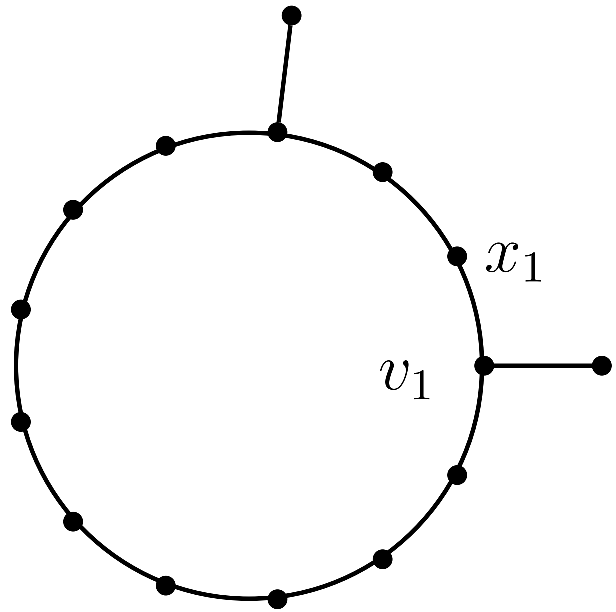

Figure 9: The sun Sun$(2, 9)$.

Avd suns with at most two rays

Clearly, every sun with one ray is avd since it is traceable.
Theorem. (R.Kalinowski, M.Pilśniak, MW and I.Zioło)
Sun(a,b) with two rays is arbitrarily vertex decomposable if and only if at most one of the numbers a and b is odd. Moreover, Sun(a,b) of order $n = a+b+4$ is not avd if and only if $(2)^{n/2}$ is the unique admissible and non-realizable sequence.

Avd suns with three rays

Theorem.
Sun(a, b, c) with three rays is not arbitrarily vertex decomposable if and only if at least one of the following three conditions is fulfilled:

(1) at least two of the numbers a, b, c are odd,

(2) $a \equiv b \equiv c \equiv 0 \pmod 3$,

(3) $a \equiv b \equiv c \equiv 2 \pmod 3$.

Examples of AVD graphs

- Of course, graphs containing a hamiltonian path

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- Of course, graphs containing a hamiltonian path
- Theorem (A. Marczyk (2005))
  If G is a two-connected graph on n vertices with the independence number at most $\lceil n/2 \rceil$ and such that the degree sum of any pair of nonadjacent vertices is at least $n-3$, then G is arbitrarily vertex decomposable with two exceptions.

Partition “on-line"

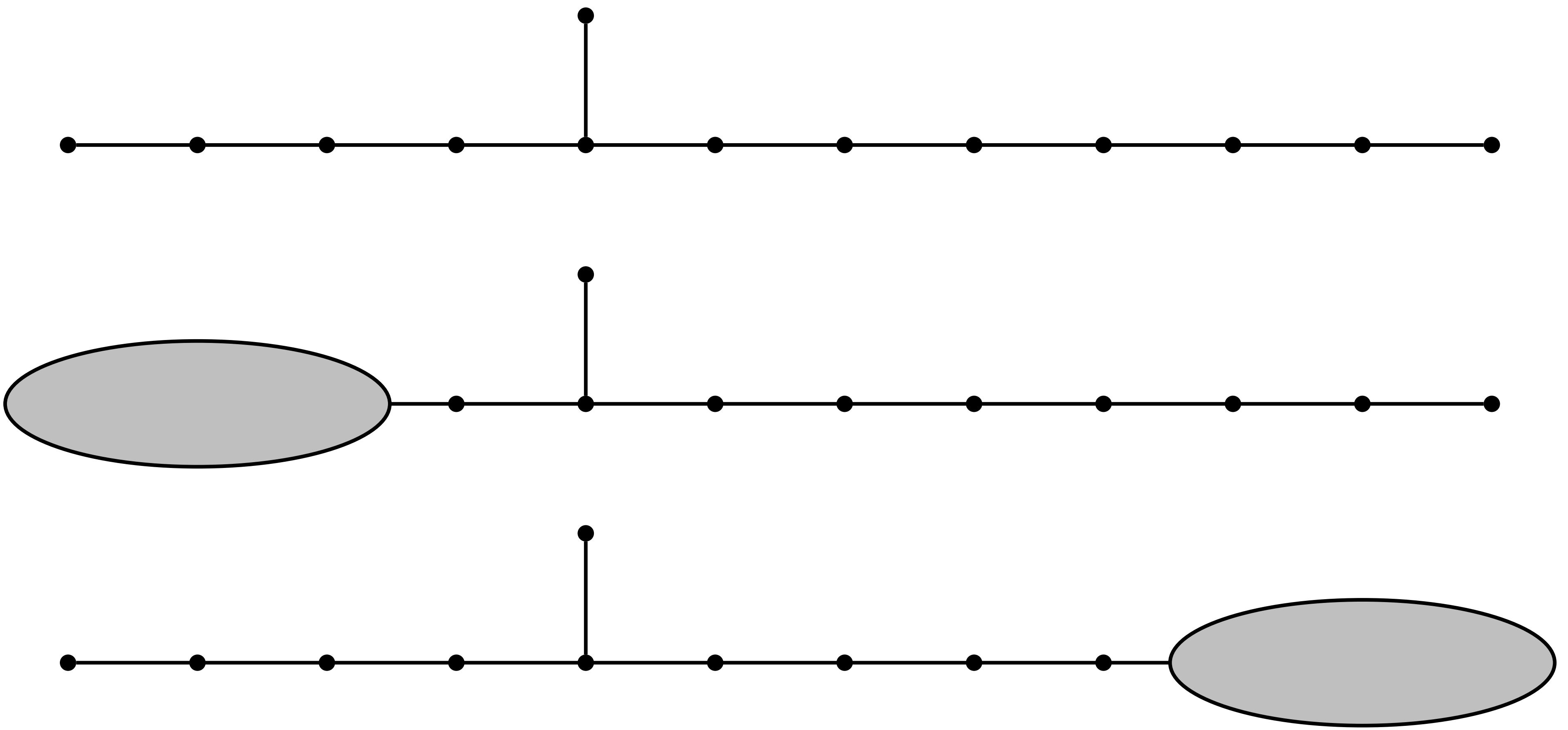

Figure 10: $\mathrm{Cat}(5,8)$

Partition “on-line"

Theorem. (Mirko Horňák, Zsolt Tuza, MW)
A tree T is AVD “on-line” iff T is either a path, or a caterpillar with one leg $\mathrm{Cat}(a, b)$, where a and b are given below or T is a tripode $S(3, 5, 7)$.

Table

a	b
2	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
4	$\equiv 1 \pmod{2}$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod{6}$
7	8, 9, 11, 13, 15
8	11, 19
9	11
10	11
11	12

Recursively AVD graphs

- Definition A graph G is said to be *recursively arbitrarily vertex decomposable* (RAVD for short) if it is AVD and for each admissible sequence there exists a realization such that the induced subgraphs $G[V_i]$ are RAVD.

Recursively AVD graphs

- Definition A graph G is said to be *recursively arbitrarily vertex decomposable* (RAVD for short) if it is AVD and for each admissible sequence there exists a realization such that the induced subgraphs $G[V_i]$ are RAVD.
- Observation An RAVD graph is on-line AVD.

AVD “on-line" but not recursively

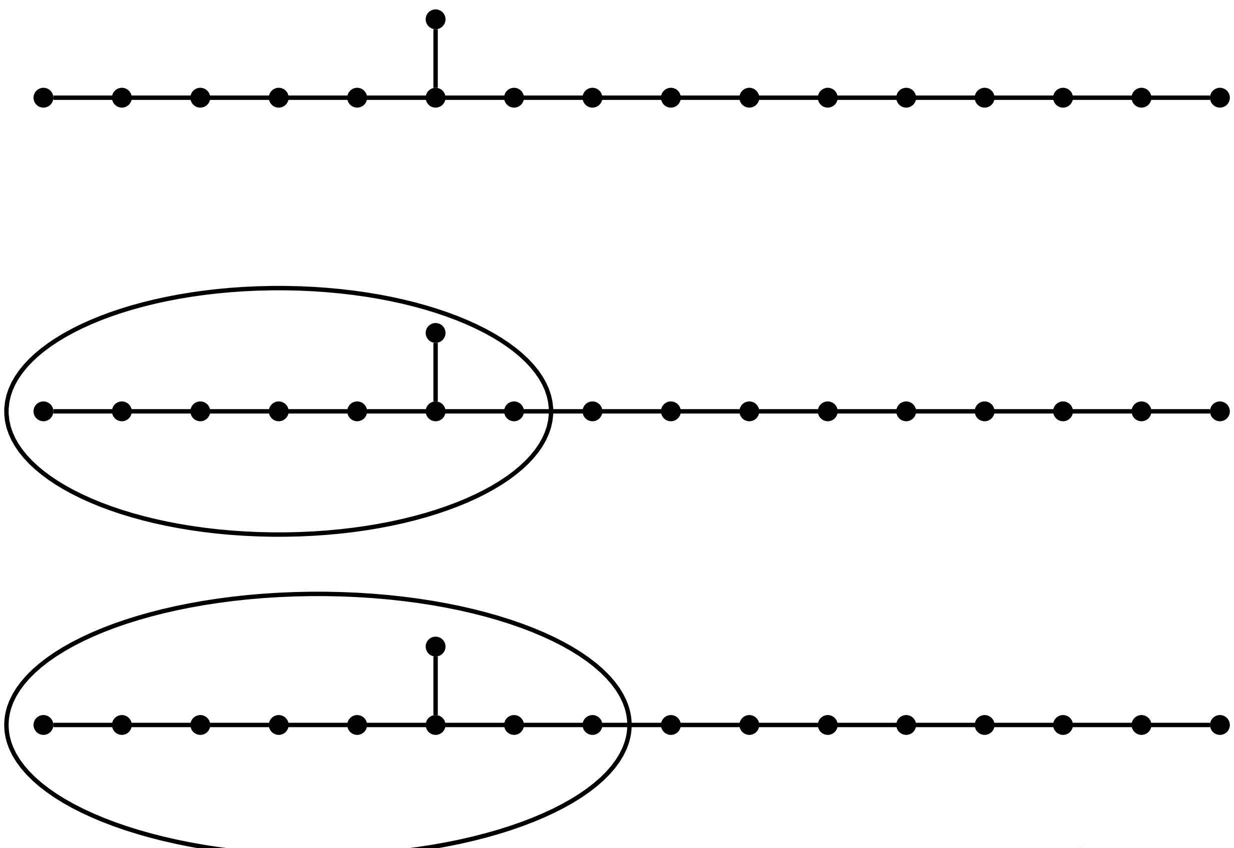

Figure 11: $\mathrm{Cat}(6, 11)$

Strongly recursively AVD graphs

- Definition A graph G is said to be *strongly recursively arbitrarily vertex decomposable* (SRAVD for short) if it is AVD and for each realization of an admissible sequence the induced subgraphs $G[V_i]$ are SRAVD.

Strongly recursively AVD graphs

- Definition A graph G is said to be *strongly recursively arbitrarily vertex decomposable* (SRAVD for short) if it is AVD and for each realization of an admissible sequence the induced subgraphs $G[V_i]$ are SRAVD.
- Observation An SRAVD graph is RAVD.

Two observations on SRAVD graphs

- An SRAVD graph is claw-free.

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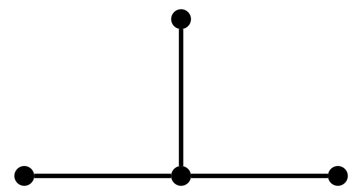

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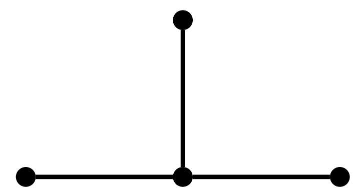

- $n = 4 + 1 + 1 + 1 + \cdots + 1$

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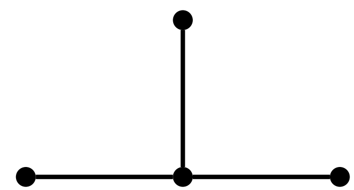

- $n = 4 + 1 + 1 + 1 + \cdots + 1$
- An SRAVD graph is net-free.

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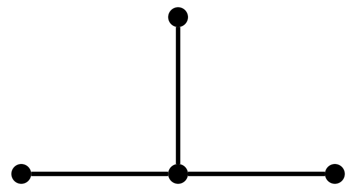

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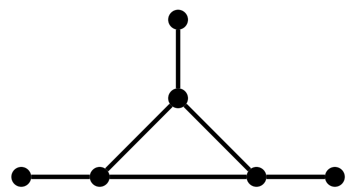

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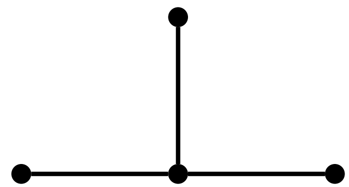

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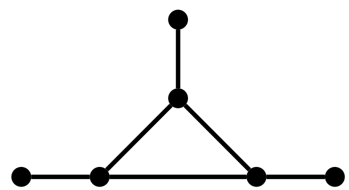

- $n = 6 + 1 + 1 + 1 + \cdots + 1$

On claw-free and net-free graphs

Theorem A connected claw-free and net-free graph is traceable.
[D.Duffus, M.S.Jacobson and R.J.Gould, Forbidden subgraphs and the hamiltonian theme (1981)]

On SRAVD graphs

Theorem(O.Baudon, MW)
A graph G is SRAVD iff G is connected and claw-free and net-free.

Some applications in real life

Decomposition of trees

Decomposition of trees. Version on-line

Thank you for your attention